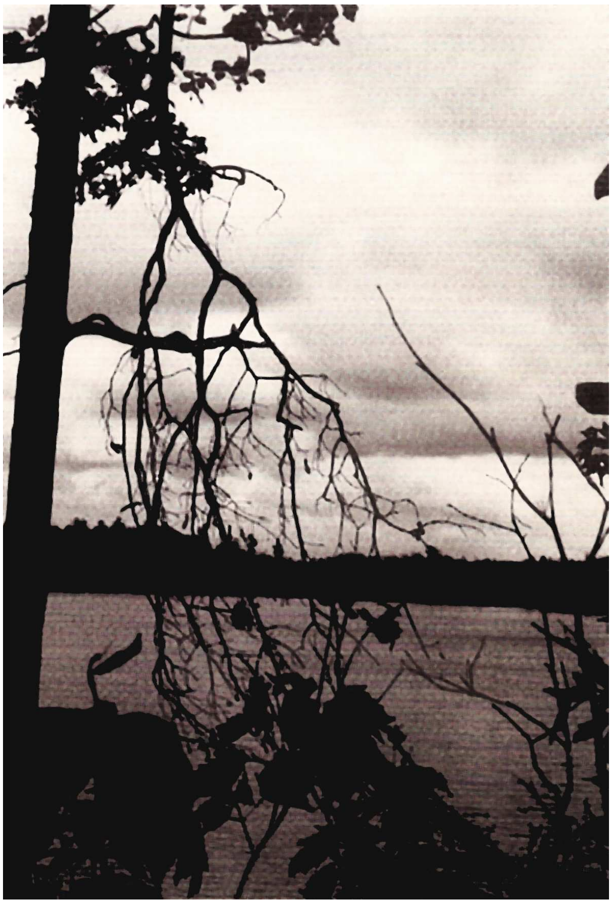
Modern intuitionistic topology



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✠ www.fwaaldijk.nl/mathematics.html

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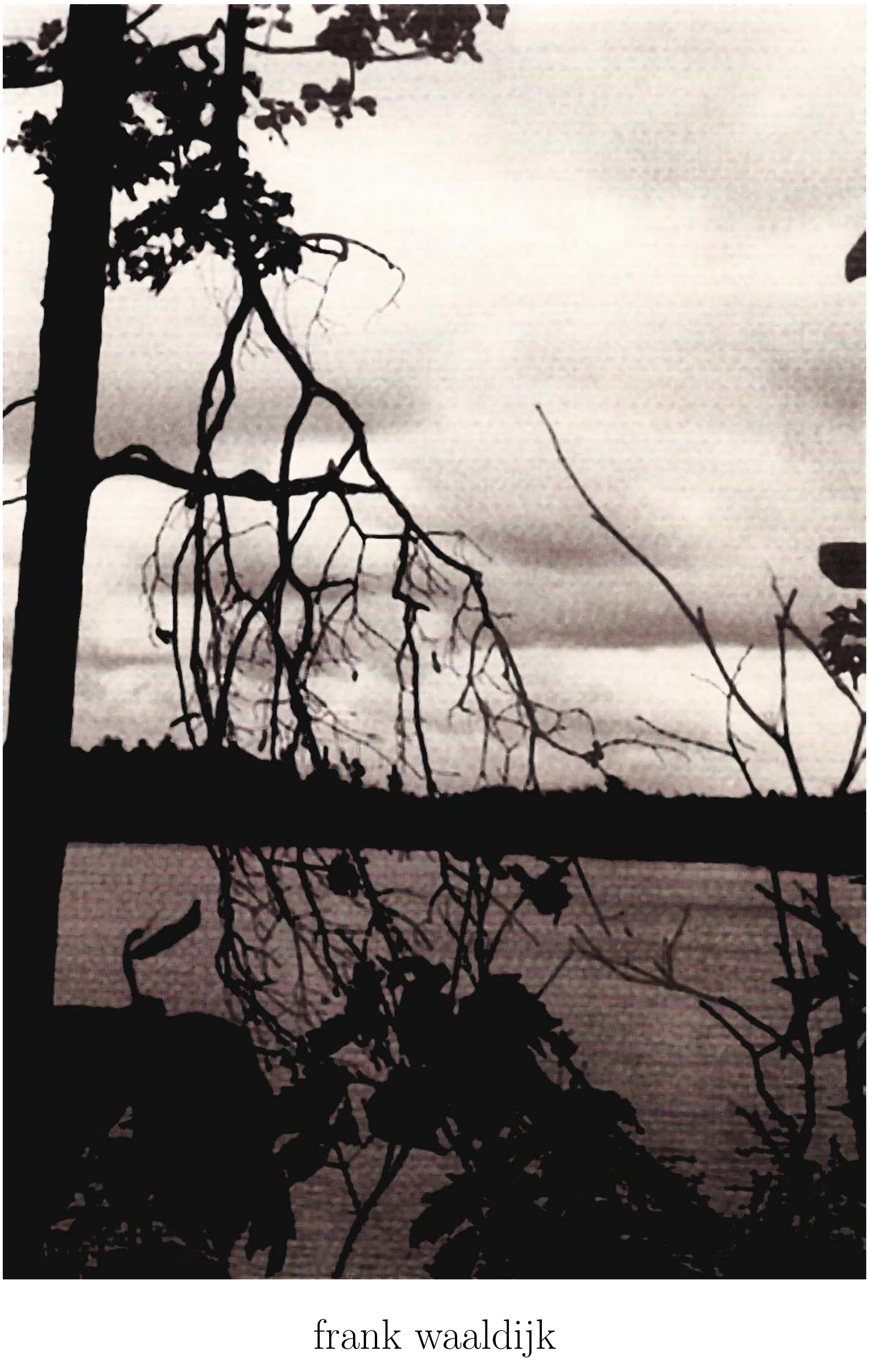

frank waaldijk

‘Ik kan ook al bijna rekenen,
ik kan mooie poppetjes tekenen.’

‘Lieve deugd’, zei de giraffe,
‘kerel, kerel, kerel, ik sta paf.’

modern intuitionistic topology

een wetenschappelijke proeve op het gebied van de Wiskunde en Informatica

PROEFSCHRIFT

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FOREWORD

The present monograph describes the last part of an adventure which started almost twelve years ago, when I first began studying mathematics in Nijmegen.

Let me compare mathematics to an ocean. Many people love to see it, standing on the beach. Many people have never seen it. Some people swim in it, and some like to snorkel around, and a handful of these end up taking diving lessons. To all people it remains a wondrous affair, with mysterious beauty and power.

Most of my diving lessons were given by Wim Veldman. He teaches the aquatic ecology of mathematics: foundations. His humble but precise style reflects the ecologist who understands the incomprehensibility of Nature, and the need to respect all life forms. He attracted me to his field of research, and convinced me not only to apply for my second solo diving project (the writing of a PhD thesis), but also to continue with it. There is little doubt in my mind that but for Wim Veldman this or a similar monograph would not have seen the light of day.

The project also brought out some differences in our characters. My style is less humble, and often lacks patience since my first wish is to explore, somewhat regardless of 'details'. Add to this the stubbornness which is necessarily inherent in *every* mathematician, and you can understand that we have seen ups and downs. The balance however was always quite positive, and I know myself well enough to blame this largely on Wim's kindness and understanding.

As my supervisor Wim had a quite direct mathematical influence on my investigations. I started out with intuitionistic model theory, which I already studied for my Master's thesis. Part of the model theoretic results are contained in a joint paper, actually written by Wim. I promised to write a follow-up article, but I still haven't found time since halfway 1993 the research drifted into intuitionistic topology. This was no coincidence. Some intuitionistic model theoretic results resemble results in classical *topological* model theory. This is understandable, since in a first-order theory of an apartness relation, it can be expressed by a first-order sentence that a first-order predicate describes an open set in the apartness topology (see chapter one) of a model.

The direct spark for the change in direction however was Wim's question whether each Σ^1_0-apartness on a spread can be weakly metrized. From this question the rest of the thesis followed. A few other questions, sometimes persistently asked, have led to considerable improvements. In chapter four for example, I had devised the extension d^* of a metric d (on a set X) only on the 'free convex hull' of X. This was in fact sufficient to prove what I wanted to prove, but I needed a large number of ad hoc amendations to the Michael theorem. A little to my annoyance Wim persisted in his belief that d could be extended to all of a linear space containing X. I found out he was right, and the result is that the Michael theorem is now used directly, saving the reader no end of trouble.

Another person who has been quite inspiring to me is Arnoud van Rooij. His courses excel in clarity and precision, and he often helped me in the beginning of my math studies. We had a few discussions on the topics in this thesis, and each of these discussions helped to clear my mind. The definition of 'compact' in chapter one occurred to me after one such discussion (and some earlier questions by Wim of course).

The switch of my investigations to topology coincided with the commencement of my spiritual training in Sahaj Marg[1], under guidance of Parthasarathi Rajagopalachari. As a side effect of the simple meditation I noticed a change in the way mathematical results came to me. I was used to a laborious thinking process, with many sideways and detours. Now ideas and even complete theorems started to occur to me while standing under the shower, or walking in the street, seemingly independent of what I was doing. And these ideas proved to be fruitful. As an example, the notion of 'halflocated' (see chapter three) occurred to me quite early, together with 'sublocated' and 'traceable'. I decided to investigate these notions, although I knew of no situation in which I could need them. It was difficult to convince Wim of their worth, and I almost gave them up. Then all of a sudden 'halflocated' started to play a central part in chapter four, since (X, d) turned out to be halflocated in (X^*, d^*), but in general not located.

At the end of 1994, hardly one and a half years later, this occurring of ideas resulted in a list of theorems and lemmas which formed the skeleton of the thesis-to-be. It contained some remarks on how I thought these lemmas and theorems could be proved, but many true proofs were still lacking. In a way this skeleton was superior to what you have before you now. It was easy to read, and a nice challenge to the do-it-myself mathematician. The main reason for not presenting the thesis in this reduced form is that it took a full year to fill in the details. One such 'detail' is chapter one, others are the section on locally compact spaces and the section on 'weak stability'. The hints and lemmas in the skeleton manuscript were often incomplete and sometimes downright wrong, but fortunately the

[1] the Natural Path

theorems have remained standing.

Wim Couwenberg and Saskia Oortwijn have been very special colleagues. In a way I was sorry to see them finish their PhD thesis ahead of me, since they brightened my time here with their enthusiastic support, which was not confined to mathematics. Also many other people of the Mathematisch Instituut contributed in one way or the other. Our local wizard Ben Polman for instance, who once wiped out an afternoon's work on the computer with the pressing of a few keys. Of course his friendly help actually saved me weeks of work, both before and after this incident. Machiel and Onno have been excellent roommates, and Trees, Willy, Hanny and Nel took care of my logistical problems. I also enjoyed playing bridge with our lunchtime bridge group.

Due to her pregnancy of Femke, our second child, Suzan has had to keep largely to her bed for the past half year or more. This time would have been impossible but for the continuous help of our friends. Especially Annemiek. And Gemma. And Marianne. They have been angels. Truus of de Stichting Thuiszorg Gelderland took care of our household and much more. A list of all the other people who helped out would be too long, but each of them can be sure that their help has been indispensable.

Nora and Femke have been my prime source of gaiety (and sleepless nights). They have had little trouble to take my mind off mathematics, thus contributing greatly to my mental health. My friends who saw little of me, my family and Suzan's family, they were there when needed.

The last and most important person who contributed to this thesis, not in words, not mathematically, but through her love, is Suzan.

the author
January 1996, Nijmegen

CONTENTS

INTRODUCTION

abstract

A brief sketch of the history of intuitionistic topology. A brief and incomplete description of Bishop-style mathematics, and some of its problems regarding topology. A brief synopsis of the contents of this thesis.

i.0 Brouwer, topology and intuitionism

i.0.0* at the cradle of both the disciplines intuitionism and topology is the work of L.E.J. Brouwer (1881-1966). We will try to sketch why it is hardly a coincidence that the founder of intuitionism was a famous topologist.

i.0.1* Brouwer expressed his views on the foundations of mathematics already in 1907 in his PhD thesis 'Over de Grondslagen der Wiskunde'[1] ([Brouwer07]). The thesis contains the philosophical outline of intuitionism, as well as a sharp critique on the philosophy of 'classical' mathematics. On the other hand [Brouwer07] also treats several problems of a strongly topological nature. The subject of topology was only just emerging at this time, but occupied some famous contemporary mathematicians, such as Cantor, Poincaré, Jordan, Peano, Hadamard, Schoenflies, Lebesgue and others.

i.0.2* Brouwer's thesis supervisor Korteweg rightly feared that Brouwer's intuitionistic views would meet with much opposition. He therefore advised Brouwer to first make a name for himself in topology, in order to secure a university position. Once having this position he would be free to pursue intuitionistic mathematics.

Brouwer followed this advice with remarkable success. He concentrated on topology between 1907 and 1913, and achieved famous results such as the invariance of dimension and domain, the Jordan curve theorem for arbitrary dimensions, the fixed point theorem, and the plane translation theorem. He also laid the grounds for dimension theory in [Brouwer13], although outside recognition of this came late. To arrive at these results he used methods which became the starting point for algebraic topology.

In October 1912 he obtained a position at the university of Amsterdam (shortly later Korteweg vacated his own position in favour of Brouwer). In his inaugural speech Brouwer returned to intuitionism. From 1913 on his publications are first and foremost on intuitionism.

i.0.3* Brouwer's mastery of topology now enables him to set up intuitionistic mathematics in a precise and rigorous way. He defines the concept of a *spread*, which corresponds to the classical notion of a closed subset of $\mathbb{N}^{\mathbb{N}}$. A *fan* is a special spread which corresponds to the classical concept of a compact subset of $\mathbb{N}^{\mathbb{N}}$. We can picture a spread as a tree, the

[1] On the Foundations of Mathematics

infinite branches of which are its elements. So the elements of a spread σ are sequences of natural numbers $\alpha = \alpha(0), \alpha(1), \ldots$. In intuitionism such a sequence arises in the course of time, as a step-by-step construction which we perform ourselves. At each step n we must choose a natural number from an inhabited decidable subset $\overline{\overline{\sigma}}(n)$ of the natural numbers which is is specified by σ, and this is the only a priori restriction. We are never done with the construction of even one element α of σ, since this construction takes infinite time. Of course we could specify beforehand that we will always choose e.g. $\alpha(n)=0$, for each $n\in\mathbb{N}$, but we need not restrict ourselves in such a way. Now a fan is a spread σ such that at each step n in the construction of an element α we have only a finite number of possible choices for $\alpha(n)$. In other words: $\overline{\overline{\sigma}}(n)$ is finite for all $n\in\mathbb{N}$. We can picture a fan as a finitely branching tree.

These concepts suffice to capture many mathematical structures of what could popularly be called *separable* classical mathematics. The set $\mathbb{R}$ of the real numbers can be built as a spread, the unit interval $[0,1]$ can be built as a fan. But also for instance the set $\mathbb{C}_p$ of the complex p-adic numbers (p a prime number) can be built as a spread, as well as $\mathbb{R}^{\mathbb{N}}$. On the other hand the space of all continuous functions from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$ (with respect to the product topology) cannot be built as a spread. Still it can be obtained as a separable subset of a spread. We will therefore in this thesis concentrate on spreads and separable subsets of spreads.

Along with the notion of a spread Brouwer arrived at what is nowadays called the *continuity principle* **CP** and 'Brouwer's principle for numbers' which we call $\mathbf{AC}_{10}$. Another insight is the *fan theorem* **FT**, which he uses together with **CP** to prove that every function from $([0,1], d_{\mathbb{R}})$ to $(\mathbb{R}, d_{\mathbb{R}})$ is uniformly continuous ([Brouwer27]). Brouwer derives the fan theorem from a more fundamental insight called the *bar theorem*, which is nowadays considered an axiom. For a precise formulation and explanation of the concepts mentioned so far, we refer the reader to chapter zero.

The concepts above are highly topological in nature. The fan theorem is an effective tool to deal with compact spaces. The continuity principle alone suffices to prove that every function from $(\mathbb{R}, d_{\mathbb{R}})$ to $(\mathbb{R}, d_{\mathbb{R}})$ is continuous (thm. 0.4.1). $\mathbf{AC}_{10}$ gives us that every spreadlike topological space with an enumerable basis is Lindelöf (thm. 1.1.6). We cannot summarize all similar results in a few lines. We hope to have given some explanation why the combination intuitionistic topology is especially rich.

i.0.4* intuitionistic topology was already studied by Brouwer. For instance Brouwer showed that his classical fixed point theorem fails intuitionistically. Also he gave an intuitionistic proof of the Jordan curve theorem. We refer the reader to [Brouwer75]. In 1937 H. Freudenthal wrote an article 'Zum intuitionistischen Raumbegriff' ([Freudenthal37]) in which he showed that Brouwer's 'katalogisiert-kompakt' spaces could be topologically characterized ('metrik-los') by intersection properties of a system of their closed subsets. Freudenthal's spaces are called DFTK-spaces. In 1966 A.S. Troelstra's PhD thesis 'Intuitionistic General Topology' ([Troelstra66]) appeared. It gives amongst others an axiomatic treatment of intuitionistic topology, with a strong emphasis on spreads. It also contains a study of 'locally DFTK-spaces'.

There is some more scattered literature. But intuitionistic topology has not kept up pace with the impressive developments in classical topology. This is probably due to the difference in the number of researchers engaged in intuitionistic and classical mathematics. Also, intuitionistic mathematics, especially intuitionistic logic, is mostly studied by logicians and computer scientists (often with classical arguments).

Some of these 'logical' studies have shed much light on fundamental issues. A beautiful example is the book 'Foundations of Intuitionistic Mathematics' by S.C. Kleene and R.E. Vesley ([Kleene&Vesley65]). In this book the above mentioned intuitionistic principles are first informally explained. Then they are expressed as axioms in a formal system. The consistency of this system is derived meta-mathematically. The independence and interdependence of the axioms is studied. It is also shown that the fan theorem is incompatible with so-called *recursive mathematics*, in which it is assumed that all sequences of natural numbers are given by a recursive rule (a computer program if you prefer).

i.1 Bishop's school

i.1.0* as mentioned, there has been comparatively little effort to fully develop other parts of intuitionistic mathematics than logic. But there is a noteworthy exception. In 1967 E. Bishop wrote a book called 'Foundations of Constructive Analysis' ([Bishop67]). In this book he rejects the classical foundations of mathematics, but also parts of intuitionism such as the continuity principle and the fan theorem. Bishop largely agrees with Brouwer's criticism of classical mathematics, but does not wish to develop a brand of mathematics which actually contradicts classical mathematics. We cannot in detail discuss Bishop's point of view (called *Bishop's school*) here. One of the main elements is that constructive mathematics should be concerned with 'constructivizing' classical mathematics, and not (too much) with the logical investigation of formal systems. There is a respectable and growing amount of research into Bishop-style mathematics.

i.1.1* a theorem in Bishop's school can always be translated to a theorem in classical mathematics; it can often be translated to a similar theorem in intuitionistic mathematics. It is also translatable to a theorem in recursive mathematics. We should however not forget that the intuitionistic interpretation of the mathematical objects involved is completely different from the classical interpretation and the recursive interpretation. Also even classical mathematicians should take heed of the definition of 'locally compact' in Bishop's school, since it differs from the classical definition (see below). Still we will develop the subject in such a way, that many parts are acceptable or easily translated in Bishop's school. We also try to follow the classical approach to topology as closely as possible (which is not always close). We discuss a few difficulties in Bishop's school which especially concern topology.

i.1.2* as said, in Bishop's school the fan theorem and the continuity principle (and therefore also $\mathbf{AC}_{10}$) are rejected. The solution for the problems that thus arise often lies in a modification of the definitions. For instance, [Bishop67] considers only continuous real functions, and leaves the question whether each real function is continuous for unsettled. However, there is a problem precisely with the definition of 'continuous function' in Bishop's school. This problem is very much related to the fan theorem.

i.1.3* in [Bishop67] a limited definition of 'continuous' is given, let us call this 'continuous$^{\text{BIS}}$'. This definition is both intuitionistically and classically equivalent to the more usual definition of a continuous function between metric spaces. Intuitionistically the fan theorem is necessary to show this equivalence. We hold: in Bishop's school it is not possible to prove that the composition of two continuous$^{\text{BIS}}$ functions is always continuous$^{\text{BIS}}$.

i.1.4* in [Bridges79] an attempt is made to remove this deficiency. Let us call this definition 'continuous$^{\text{BRI}}$', which also is intuitionistically and classically equivalent to the more usual definition. The composition of two continuous$^{\text{BRI}}$ functions is again continuous$^{\text{BRI}}$. But let h be the bijection from $(\mathbb{R}^+, d_{\mathbb{R}})$ to $(\mathbb{R}, d_{\mathbb{R}})$ which is completely determined by:

$$h(x) \underset{D}{=} \begin{cases} 2 - \frac{1}{x} & \text{for } x \leq 1 \\ x & \text{for } x \geq 1 \end{cases}$$

We hold: in Bishop's school it is not possible to prove that both h and h^{-1} are continuous$^{\text{BRI}}$. (On the other hand, it is easily seen that h and h^{-1} are continuous$^{\text{BIS}}$).

i.1.5* the remarks in i.1.3 and i.1.4 are proved thus. Since Bishop-style mathematics is compatible with recursive mathematics, it is not possible in Bishop's school to prove that every continuous$^{\text{BIS}}$ function f from $([0,1], d_{\mathbb{R}})$ to $(\mathbb{R}^+, d_{\mathbb{R}})$ is bounded away from 0 (meaning: $\exists \epsilon \in \mathbb{R}^+ \; \forall x \in [0,1] \; [\, f(x) > \epsilon \,]$). For in recursive mathematics there is an example of a continuous$^{\text{BIS}}$ f from $([0,1], d_{\mathbb{R}})$ to $(\mathbb{R}^+, d_{\mathbb{R}})$ such that $\forall \epsilon \in \mathbb{R}^+ \; \exists x \in [0,1] \; [\, f(x) < \epsilon \,]$, see [Beeson85, thm.IV.8.1].

On the other hand, if g is a continuous$^{\text{BIS}}$ or a continuous$^{\text{BRI}}$ function from $([0,1], d_{\mathbb{R}})$ to $(\mathbb{R}, d_{\mathbb{R}})$, then it is possible in Bishop's school to compute both $\sup(\{g(x) \mid x \in [0,1]\})$ and $\inf(\{g(x) \mid x \in [0,1]\})$.

Now suppose that in Bishop's school the composition of two continuous$^{\text{BIS}}$ functions is always continuous$^{\text{BIS}}$. Let f be a continuous$^{\text{BIS}}$ function from $([0,1], d_{\mathbb{R}})$ to $(\mathbb{R}^+, d_{\mathbb{R}})$. Then by assumption the composition $g = h \circ f$ is a continuous$^{\text{BIS}}$ function from $([0,1], d_{\mathbb{R}})$ to $(\mathbb{R}, d_{\mathbb{R}})$. So we can compute $\inf(\{g(x) \mid x \in [0,1]\})$. But clearly then the function $h^{-1} \circ g$, which is f, is bounded away from 0. So then we would obtain in Bishop's school that every continuous$^{\text{BIS}}$ function f from $([0,1], d_{\mathbb{R}})$ to $(\mathbb{R}^+, d_{\mathbb{R}})$ is bounded away from 0, which is impossible by our remark above. Precisely the same argument of course gives that in Bishop's school h and h^{-1} cannot both be shown to be continuous$^{\text{BRI}}$.

i.1.6* this difficulty has its impact on the definition of 'locally compact' in Bishop's school. Both in intuitionism and in classical mathematics the function h is a homeomorphism. In Bishop's school $(\mathbb{R}, d_{\mathbb{R}})$ is locally compact, but $(\mathbb{R}^+, d_{\mathbb{R}})$ is not locally compact. In the context of topology this approach cannot be sustained.

i.1.7* another consequence of the difficulty described above is the following. In [Bishop67], [Bridges79] and [Bishop&Bridges85] a constructive limited version of the so-called Tietze extension theorem is proved. The version runs as follows (with Bishop-style definitions of 'locally compact' and 'continuous'):

theorem: let (A, d) be a locally compact subspace of a metric space (X, d). Let f be a continuous function from (A, d) to $([0,1], d_{\mathbb{R}})$. Then there is a continuous extension of f to (X, d).

(Notice that without translation the theorem is classically false.). It is also said that this formulation is as much as what is true constructively of the classical Tietze theorem. However we will *constructively* prove a version of the Dugundji Extension Theorem (which implies the Tietze theorem) in which the above condition 'locally compact' is brought back to 'strongly halflocated', see 4.1.1. 'Strongly halflocated' can be seen as a constructive formulation of the classical condition 'closed' in the Dugundji theorem. The only problem with this constructive version of the Dugundji theorem is the following. It is perhaps difficult to prove in Bishop's school that the extension found is continuous$^{\mathrm{BIS}}$ and/or continuous$^{\mathrm{BRI}}$.

i.1.8* the fan theorem solves the above mentioned problems in a simple way, and we do not see an easy other solution. There is one more problem which we mention. In Bishop's school some very liberal axioms of choice are used, in which there is no a priori restriction on the 'domain' and the 'range' of the choice functions. We cannot discuss this at length here. The reader may consult [Troelstra&vanDalen88, sect.4.2]. We emphasize that our axioms of choice (see chapter zero) are for spreads only. Therefore the domain and range of our choice functions are limited to spreads. Perhaps the restriction on the range can be relaxed a little in the case of 'countable choice' and 'dependent choice'. But we are definitely not convinced of the validity of choice axioms for 'arbitrary domains' and 'arbitrary ranges'. The following example shows that a too liberal axiom of choice causes constructive problems. (See also [Troelstra&vanDalen88, sect.4.2]).

EXAMPLE: let $\alpha_{k_{99}}$ be the sequence of natural numbers less than 2 given by: $\alpha_{k_{99}}(n)=1$ if and only if n is the first 9 in the first block of 99 consecutive 9's in the decimal expansion of π (for $n\in\mathbb{N}$). Let $\underline{0}$ be the sequence of natural numbers given by: $\underline{0}(n)=0$ for all $n\in\mathbb{N}$. Let D be the subset of $\mathbb{N}^{\mathbb{N}}$ given by: $D=\{\underline{0},\alpha_{k_{99}}\}$. Let A be the subset of $D\times\mathbb{N}$ given by: $A=\{(\underline{0},0),(\alpha_{k_{99}},1)\}$. Clearly we have:

$(\star)$ $\forall\alpha\in D\ \exists s\in\{0,1\}\ [\,(\alpha,s)\in A\,]$

But the existence of a choice *function* for $(\star)$ would enable us to decide whether there occurs a block of 99 consecutive 9's in the decimal expansion of π, or not. We believe that no one as of yet has come up with a method to make this decision.

i.1.9* let us now proceed to the topology in this thesis. We must first of all admit that the present monograph hardly narrows the gap in development between classical and intuitionistic topology. But the impressive body of classical topology can well be compared to Goliath; and we are practising our sling.

i.2 SYNOPSIS OF CHAPTER ONE

i.2.0* in the first section of chapter one we study the idea of a 'topological space', notation $(X,\mathcal{T})$ (where X is a subset of $\mathbb{N}^{\mathbb{N}}$ and $\mathcal{T}$ is a topology on X). Our analysis leads us to impose certain constructive restrictions on $(X,\mathcal{T})$. First of all we exact that the topology $\mathcal{T}$ be *effective*. This means that if U is in $\mathcal{T}$ and x is an element of U, then for all y in X we can decide:

$y \in U$, or there is a V in $\mathcal{T}$ such that $x \in V$ and $y \notin V$.

If $(X,\mathcal{T})$ is an effective topological space, then $\mathcal{T}$ induces an apartness $\#_{\mathcal{T}}$ on X by setting: $x \#_{\mathcal{T}} y$ if and only if there is a U in $\mathcal{T}$ such that: $x \in U$ and $y \notin U$, or $y \in U$ and $x \notin U$. We also demand that $\#_{\mathcal{T}}$ is a Σ^1_0-apartness. This means that for all x and y in X: the relation $x \#_{\mathcal{T}} y$ is decided in the course of time. More precise, for every x and y in X there is a sequence δ of natural numbers less than 2, such that $x \#_{\mathcal{T}} y$ if and only if there is an $n \in \mathbb{N}$ with $\delta(n)=1$. For a very large class of topological spaces the topology is effective and the induced apartness is a Σ^1_0-apartness.

On the other hand, we discover that every apartness $\#$ on X induces an effective topology $\mathcal{T}_{\#}$ on X, called the $\#$-topology or the apartness topology. A subset U of X is in $\mathcal{T}_{\#}$ if and only if for each x in U and each y in X: $y \in U$ or $x \# y$. We simply write $(X, \#)$ for this *apartness space*.

So given a topological space $(X,\mathcal{T})$, we obtain an apartness $\#_{\mathcal{T}}$. An important yet easy result is that the $\#_{\mathcal{T}}$-topology refines $\mathcal{T}$. The apartness topology plays a fundamental part in our account. Many important spaces are in fact apartness spaces. For instance every topologically complete space is an apartness space, see chapter three.

Our final restriction is that $(X,\mathcal{T})$ be second-separable, meaning that there is a sequence $(x_n)_{n\in\mathbb{N}}$ in X which is dense in $(X,\mathcal{T})$. (A first-separable topological space is a space $(X,\mathcal{T})$ with an enumerable basis for $\mathcal{T}$).

i.2.1* in the second section a number of general topological concepts are defined, such as 'continuous function', 'open cover', 'spreadlike' (resp. 'fanlike'), 'Lindelöf', 'connected', and so on. Every function from an apartness space to another topological space is continuous. A first-separable spreadlike $(X,\mathcal{T})$ is seen to be Lindelöf. We constructively define the well-known classical separation properties 'T_1', 'Hausdorff', 'regular' and 'normal'.

i.2.2* the third section deals with 'compact' spaces. A topological space $(X,\mathcal{T})$ is defined to be compact if and only if $(X,\mathcal{T})$ is fanlike and Hausdorff. This closely parallels the classical definition. Every compact space is an apartness space, therefore every function from a compact space to another topological space is continuous. Every compact space will be shown to be metrizable in section 2.1, but there are many compact spaces which are not topologically complete. A topologically complete compact space is called *strongly compact*.

i.2.3* in the fourth section we study subspaces with the *subspace topology*. An important concept already found in [Troelstra66] is that of a *sublocated* subspace $(A,\mathcal{T}_A)$ of a topological space $(X,\mathcal{T})$. Classically this is an empty condition, but intuitionistically it provides the necessary minimum information on the 'whereabouts' of A. We define a stronger concept 'strongly sublocated in', which classically would be equivalent to 'closed'. It serves as a strong intuitionistic analogon of 'closed'. The property 'strongly sublocated in' behaves transitively, the property 'sublocated in' does not. A subspace $(A,\mathcal{T}_A)$ of an apartness space $(X,\mathcal{T})$ is strongly sublocated in $(X,\mathcal{T})$ iff for all x in X there is a y in A such that: $x\#y$ implies $x\#a$ for all a in A.

i.2.4* the fifth section is concerned with the following question. If 'C' is a topological concept, then what should it mean for a topological space to be 'locally C'? We follow the classical approach: $(X,\mathcal{T})$ is 'locally C' if and only if for each x in X and each $U\ni x$ in $\mathcal{T}$ there is a neighborhood W of x in $(X,\mathcal{T})$ such that: $W\subseteq U$ and $(W,\mathcal{T}_W)$ is a topological space which is 'C'. This convention is most manageable in the case where $(X,\mathcal{T})$ is spreadlike; we then say that $(X,\mathcal{T})$ is '1-locally C'. In this way the definitions of '(1-)locally compact', '(1-)locally connected', and so on, are obtained. Every locally compact space is an apartness space, therefore every function from a locally compact space to another topological space is continuous.

i.3 SYNOPSIS OF CHAPTER TWO

i.3.0 in the first section of chapter two we introduce the notion of a *touch-relation*. Let σ be a spread, then write $\overline{\overline{\sigma}}$ for the decidable subset of $\mathbb{N}$ which contains precisely the (encodings) of the finite initial segments of the infinite sequences in σ. A touch-relation $\approx$ on $\overline{\overline{\sigma}}$ is a decidable symmetric and reflexive subset of $\overline{\overline{\sigma}}\times\overline{\overline{\sigma}}$ such that the complement $\not\approx$ induces a Σ^1_0-apartness by putting: $\alpha\#\beta$ if and only if $\exists n\in\mathbb{N}\,[\overline{\alpha}(n)\not\approx\overline{\beta}(n)]$. We prove: every Σ^1_0-apartness on a spread σ is induced by a touch-relation on $\overline{\overline{\sigma}}$.

An example is given of a $(\sigma, \#)$ which is T_1 but not Hausdorff, and of a $(\sigma, \#)$ which is Hausdorff but not regular.

i.3.1 in the second section we prove that every apartness fan is metrizable. As a corollary we obtain that every compact space is metrizable. In fact we see that a space $(X, \mathcal{T})$ is compact iff it coincides with an apartness fan, iff it coincides with a metric fan.

The proof of the above metrizability theorem leads us to an example of a compact pathwise connected and locally pathwise connected space which is not arcwise connected and not locally arcwise connected.

i.3.2 the third section is a detailed analysis of 1-locally compact spaces. A space $(X, \mathcal{T})$ is 1-locally compact iff it admits an enumerable cover with compact neighborhoods iff it has a one-point compact extension. A compact extension of $(X, \mathcal{T})$ is called a compactification of $(X, \mathcal{T})$ iff $(X, \mathcal{T})$ coincides with one of its *dense* subspaces. (So contrary to popular belief not every locally compact space has a one-point compactification; consider for instance a compact space). Every 1-locally compact space is metrizable. A space $(X, \mathcal{T})$ is 1-locally strongly compact iff it has a one-point compact extension which is strongly compact iff $(X, \mathcal{T})$ is 1-locally compact and topologically complete.

i.3.3 in the fourth section we define a topological space $(X, \mathcal{T})$ to be *sigma-compact* iff there is a sequence $((W_n, \mathcal{T}_{W_n}))_{n\in\mathbb{N}}$ of compact subspaces such that $X = \bigcup_{n\in\mathbb{N}} W_n$. We give an example of a sigma-compact metric space which is not an apartness space. Sigma-compact apartness spaces are of special interest. A topological space is a sigma-compact apartness space iff it is the inductive limit of a sequence of increasing compact subspaces. Every sigma-compact space $(X, \mathcal{T})$ is weakly metrizable (meaning there is a metric d on X such that for all x, y in X: $x \#_{\mathcal{T}} y$ iff $d(x,y) > 0$). Not every sigma-compact apartness space is metrizable.

i.3.4 in the fifth section we introduce the notion of a *star-finitary* space. This is a broad generalization of the concept of '1-locally compact'. The topological product of a sequence of star-finitary spaces is again star-finitary. Every star-finitary space is metrizable. We also define a weaker concept called 'weakly star-finitary' which classically would be equivalent to 'star-finitary'.

A simple lemma shows that the Hilbert cube $(\mathcal{Q}, d_{\mathcal{Q}})$ is a compact extension of every metric space.

i.4 SYNOPSIS OF CHAPTER THREE

i.4.0* the first section contains the well-known result that every complete metric space is spreadlike.

i.4.1* in the second section we study open covers of a metric space (X, d). A *per-enumerable* open cover $\mathcal{U}$ of (X, d) has a *star-finite* refinement and a subordinate *partition of unity*. Every open cover of a spreadlike (X, d) has a per-enumerable refinement. As a consequence every spreadlike metric space is normal. Also every complete metric space is weakly star-finitary. Not every complete metric space is star-finitary.

i.4.2* the third section is an investigation of different concepts of 'locatedness' of subspaces of a metric space. In decreasing order of strength we study: best approximable, (strongly) located, (strongly) halflocated, (strongly) sublocated and (strongly) traceable in.

'(Strongly) sublocated in' and '(strongly) traceable in' are the only topological notions in this list. We define a subspace (A, d) of a metric space (X, d) to be topologically best approximable (resp. topologically (strongly) (half)located) in (X, d) iff there is a d-equivalent metric d' on X such that (A, d') is best approximable (resp. (strongly) (half)located) in (X, d').

'(Strongly) halflocated in' behaves transitively, but 'located in', 'strongly located in' and 'best approximable in' do not. A subspace (A, d) is (half)located in a metric space (X, d) iff the completion $\overline{(A, d)}$ is strongly (half)located in the completion $\overline{(X, d)}$. This property fails in general for 'sublocated in' and 'traceable in'.

If (A, d) is strongly traceable in a spreadlike (resp. compact) (X, d), then (A, d) is spreadlike (resp. compact).

If (A, d) is traceable in a compact (X, d), then (A, d) is located in (X, d). We give a Brouwerian example of a strongly sublocated subspace (A, d) of a compact (X, d) which is not strongly halflocated in (X, d). But a subspace (A, d) of a 1-locally strongly compact (X, d) is (strongly) traceable in (X, d) iff (A, d) is topologically (strongly) located in (X, d).

i.4.3* the last section is a study of the notion of 'weak stability'. A metric space (X, d) is said to be *weakly stable* iff for all y in $\overline{(X, d)}$: $\exists x \in X\ [\, y \# x \rightarrow y \in X\,]$ implies $y \in X$. 'Weakly stable' is a topological property. Every metric space (X, d) has a weakly stable closure $\overline{(X, d)}$. Every continuous function from (X, d) to another metric space (Y, d_Y) can be extended to a continuous function from $\overline{(X, d)}$ to $\overline{(Y, d_Y)}$. The weakly stable closure of a spreadlike (X, d) is again spreadlike. We prove the so-called 'Weakly Stable Continuity Principle' $\mathbf{CP}_{\mathrm{ws}}$, which implies that every weakly stable spreadlike metric space is an apartness space. This is a strong generalization of Brouwer's theorem that every function from $(\mathbb{R}, d_{\mathbb{R}})$ to $(\mathbb{R}, d_{\mathbb{R}})$ is continuous.

i.4.4* the above sections are all indispensable for our investigations in chapter four. Chapter three may therefore be seen as a tool-box for chapter four, although it is of independent interest.

i.5 SYNOPSIS OF CHAPTER FOUR

i.5.0* in the first section we define *absolute retracts* and *absolute extensors*. Our definition is quite close to the classical definition but contains 'strongly halflocated' rather than 'closed'. Every absolute extensor is an absolute retract.

i.5.1* in the second section we prove a constructive version of the Dugundji Extension Theorem. As a consequence of this theorem every weakly stable convex subspace of a locally convex linear space is an absolute extensor.

i.5.2* the third section shows how to construct, for a given metric space (X, d), a normed linear space (X^*, d^*) such that (X, d) is halflocated in (X^*, d^*). This construction could be of classical interest as well. (X, d) is strongly halflocated in (X^*, d^*) whenever (X, d) is weakly stable. This gives that a weakly stable metric space (X, d) is an absolute extensor iff it is an absolute retract iff it is a retract of a weakly stable convex subspace of a locally convex linear space.

i.5.3 in the fourth section we prove an intuitionistic version of the Michael Selection Theorem. One difficulty in proving this theorem lies in finding a partition of unity subordinate to an arbitrary open cover of a metric space (X, d). Therefore we limit ourselves to spreadlike spaces. The Michael Selection Theorem is the key to the following sections.

i.5.4 the fifth section contains a discussion of the Michael theorem in Bishop's school. This brings us to the already existing concept of 'strong continuity'. A function is *strongly continuous* iff it has a continuous modulus of continuity. The Michael theorem implies that every continuous function from a metric spread to another metric space is strongly continuous.

i.5.5 in the last section we combine a large number of previous results to arrive at the following fundamental theorem. Let (A, d) be weakly stable and strongly sublocated in a spreadlike metric space (X, d). Then (A, d) is topologically strongly halflocated in (X, d). Important steps in the proof are the following observations. First: if (A, d) is a retract of a metric space (X, d), then (A, d) is topologically best approximable in (X, d). Second: if (A, d) is strongly sublocated in a spreadlike metric space (X, d), then $\overline{(A^*, d^*)}$ is a retract of $(X \cup \overline{A^*}, d^*)$. Some variations of the theorem are given. We also obtain a stronger version of the Dugundji theorem for spreadlike spaces.

CHAPTER ZERO

PRELIMINARIES

abstract

Fixing notations first, basic intuitionistic concepts and axioms are formulated. Introduction of the basic important structures in this book. Elementary theorems and lemmas, with proofs omitted whenever they are common knowledge.

0.0 ELEMENTARY INTUITIONISM

0.0.0* much has been said and written already on the foundations of intuitionism. Instead of going into lengthy discussions, we will therefore give a brief exposition of the fundamental principles which are used in this thesis. The reader may consult [Brouwer75], [Heyting56], [Kleene&Vesley65], [Troelstra&vanDalen88] and [Veldman85] for a more complete approach. Other references for this chapter are [Bishop67], [Bridges79] and [Bishop&Bridges85], although their standpoint is not intuitionistic. We will speak of these references as 'Bishop's school' or 'Bishop-style mathematics'. Our exposition borrows freely from [Veldman85].

The heart of intuitionism lies in our intuition of time. From this intuition the *natural numbers* 0, 1, 2, ... are born, one after the other, in a never-ending process. We then construct the set $\mathbb{N}$ of all natural numbers. It should be emphasized that the construction of $\mathbb{N}$ as a whole is never finished. We construct from $\mathbb{N}$ the set $\mathbb{Z}$ of all *integers*, as well as the set $\mathbb{Q}$ of all *rational numbers*, in the usual way. We also assume that the reader is familiar with the usual operations $+$, $-$, $\cdot$ and the natural order $<$ on $\mathbb{Q}$. The *entier* of a rational number a, notation $[a]$, is the $<$-largest integer m such that $m \leq a$.

We call 0, 1, 2, ... a *sequence* of natural numbers. There are many other such sequences of course, for instance the sequence of prime numbers 2, 3, 5, ... The set of all sequences of natural numbers is often called $\mathbb{N}^{\mathbb{N}}$. In intuitionism the tradition is however to call this set σ_ω, for reasons which will become apparent in 0.0.3 and 0.0.5.

There is no way to produce all sequences of natural numbers one after the other. This is a lesson taught by Cantor's diagonal argument, exhibiting an important difference between σ_ω and $\mathbb{N}$. For we do have a way to produce all natural numbers, one after the other, even if we are never done with $\mathbb{N}$ as a whole. But to produce just one element of σ_ω is as much work as producing all of $\mathbb{N}$.

We think of an element α of σ_ω, that is a sequence of natural numbers $\alpha(0), \alpha(1), \alpha(2) \ldots$, as constructed step by step, in the course of time. At each stage n in the construction of α, we are completely free to choose the natural number $\alpha(n)$. There need not be any deterministic law or algorithm which α must comply with. On the other hand we are free to follow any such law for as long as we like. For instance, we consider it possible that the sequence $\underline{0}$ given by 0, 0, 0, etc. is the outcome of a step-by-step construction. We say that two sequences α and β in σ_ω are *equal*, notation $\alpha = \beta$, iff for all $n \in \mathbb{N}$: $\alpha(n) = \beta(n)$.

Then σ_ω is the set of all such step-by-step constructions of sequences of natural numbers. In a way σ_ω is determined by the tree $\mathbb{N}^*$ of all finite sequences of natural numbers. To be precise: $\mathbb{N}^* \underset{D}{=} \bigcup_{n\in\mathbb{N}} \mathbb{N}^{n+1} \cup \{-1\}$, where -1 stands for the empty sequence. The elements of σ_ω are the infinite walks that we go for, following the tree upwards.

We picture $\mathbb{N}^*$ as a tree, and σ_ω as the set of its infinite branches. There are other similar trees, for instance the finitely branching tree $\{0,1\}^* \underset{D}{=} \bigcup_{n\in\mathbb{N}} \{0,1\}^{n+1} \cup \{-1\}$. The infinite branches of $\{0,1\}^*$ are the sequences α in σ_ω such that for each $n\in\mathbb{N}$: $\alpha(n)<2$. The set of all such sequences is called σ_2.

0.0.1* before we continue, let us state that certain mathematical notions will be taken as *primitive*, that is: hopefully understood but not defined in terms of other notions. One of these notions is the notion of a *sequence*, for instance a sequence of natural numbers as discussed above. Another such notion is the notion of a 'subset of σ_ω' and an 'element' of a subset, along with the notion of a 'collection of subsets'. We write $\emptyset$ for the empty subset of σ_ω. Also the notion of 'method' is primitive, and it ties in with the primitive notion of 'existence'. We say that a mathematical object such as a natural number, a sequence of natural numbers, or a subset of σ_ω with a certain property P *exists* if and only if we have a method to construct it. Then we write, for example: $\exists\ \alpha\in\sigma_\omega\ [\,\mathrm{P}(\alpha)\,]$ (*there is* an α in σ_ω with property P). If P is a property applicable to sequences of natural numbers, then we can form the subset $\{\alpha\in\sigma_\omega \mid \mathrm{P}(\alpha)\}$ of σ_ω. If P is a property applicable to natural numbers and n is in $\mathbb{N}$, then we write $n{=}\mu s\in\mathbb{N}\ [\,P(s)\,]$ to mean that n is the smallest natural number with property P.

From now on we abbreviate 'if and only if' with 'iff'.

We assume the reader is familiar with the logical symbols $\forall, \exists, \exists!, \wedge, \vee, \neg$ and $\rightarrow$. We often use them to abbreviate otherwise lengthy statements. Let P be a property applicable to the elements of a set or collection X. '$\forall x\in X\ [\,P(x)\,]$' means: for all x in X we can prove $P(x)$. '$\exists x\in X\ [\,P(x)\,]$' means: there is an x in X such that $P(x)$, as explained above. '$\exists! x\in X\ [\,P(x)\,]$' means: there is an x in X such that $P(x)$ and for all y in X: if $P(y)$ then $y{=}x$. If on the other hand P and Q are statements, then '$P \wedge Q$' means: we can prove both P and Q. '$P \vee Q$' means: we can choose either P or Q, and then prove the chosen statement. So in fact '$P \vee Q$' is the same as: '$\exists s\in\{0,1\}\ [\,(s{=}0 \wedge P)$ or $(s{=}1 \wedge Q)\,]$'. '$P \rightarrow Q$' means: P implies Q (we can prove Q from P). Finally, '$\neg P$' means that we can prove a mathematical contradiction from P (and our axioms). We more frequently write 'NOT P' instead of '$\neg P$'.

We have to distinguish between P and $\neg\neg P$. Of course $\neg\neg P$ follows from P, but in general the knowledge that NOT P is impossible does not supply us with a proof of P. Similarly we distinguish between $\neg\forall x\in X\ [P(x)]$ and $\exists x\in X\ [\neg P(x)]$. There are situations in which we can prove both $\neg\forall x\in X\ [P(x)]$ and $\neg\exists x\in X\ [\neg P(x)]$.

0.0.2* we return to the discussion in 0.0.0. The tree-notion mentioned there is beautifully captured by the intuitionistic idea of a *spread*, which at the same time does full justice to the time element involved in its construction. For the definition of a spread we need some simple machinery.

DEFINITION: we fix a bijection $\langle\!\langle\rangle\!\rangle$ from $\mathbb{N}^*$ to $\mathbb{N}$ with the property that if $(a_1,\ldots,a_n)$ is a finite sequence, and b is a finite sequence beginning with $(a_1,\ldots,a_n)$, then $\langle\!\langle(a_1,\ldots,a_n)\rangle\!\rangle \leq \langle\!\langle b\rangle\!\rangle$. We will mostly write $\langle\!\langle a_0,a_1,\ldots,a_n\rangle\!\rangle$, omitting the parentheses. Also we mostly write $\langle\!\langle\rangle\!\rangle$ instead of 0 for the encoding of the empty sequence.

Let A be a subset of σ_ω. We say that A is *finite* iff there is an $n\in\mathbb{N}$ such that A contains precisely n elements. We say that A is *inhabited* iff there is an element in A, that is: $\exists\alpha\in\sigma_\omega\ [\alpha\in A]$.

Now let a be in $\mathbb{N}$. Then we write $lg(a)$ for the length of the finite sequence which is encoded by a. For $i<lg(a)$ we then write a_i for the i^{th} element of this finite sequence. Suppose $a=\langle\!\langle a_0,a_1,\ldots,a_{lg(a)-1}\rangle\!\rangle$ and $b=\langle\!\langle b_0,b_1,\ldots,b_{lg(b)-1}\rangle\!\rangle$ are in $\mathbb{N}$ then we write $a\star b$ for the concatenation $\langle\!\langle a_0,a_1,\ldots,a_{lg(a)-1},b_0,b_1,\ldots,b_{lg(b)-1}\rangle\!\rangle$ of a and b. We write $a\sqsubseteq b$ iff there is a c in $\mathbb{N}$ such that $b=a\star c$, and we write $a\sqsubset b$ iff in addition $lg(b)>lg(a)$.

Finally, let α be an element of σ_ω and let $n\in\mathbb{N}$. We write $\overline{\alpha}(n)$ for the encoding $\langle\!\langle\alpha(0),\alpha(1),\ldots,\alpha(n-1)\rangle\!\rangle$ of the first n values of α. We write $\alpha_{[n]}$ for the sequence β in σ_ω given by: $\beta(m)=\alpha(\langle\!\langle n\rangle\!\rangle\star m)$.

0.0.3* DEFINITION: let σ be an element of σ_2. We say that σ is a *spread-law* iff

(i) $\sigma(\langle\!\langle\rangle\!\rangle)=0$.

(ii) for all a in $\mathbb{N}$: $\sigma(a)=0$ iff there is an $m\in\mathbb{N}$ such that $\sigma(a\star\langle\!\langle m\rangle\!\rangle)=0$.

If σ is a spread-law, then the subset $\{\alpha\in\sigma_\omega \mid \forall n\in\mathbb{N}\ [\sigma(\overline{\alpha}(n))=0]\}$ is called a *spread*. We will also write σ for this subset. We write $\overline{\overline{\sigma}}(n)$ for the decidable subset $\{\overline{\alpha}(n) \mid \alpha\in\sigma\}$ of $\mathbb{N}$. We write $\overline{\overline{\sigma}}$ for $\{\overline{\alpha}(n) \mid \alpha\in\sigma, n\in\mathbb{N}\}$, which is equal to $\{a\in\mathbb{N} \mid \sigma(a)=0\}$. Now let a be in $\overline{\overline{\sigma}}$. We write $\sigma\cap a$ for the subspread $\{\alpha\in\sigma \mid \overline{\alpha}(lg(a))=a\}$ of σ. A spread τ is called a *fan* iff for all $n\in\mathbb{N}$ the set $\overline{\overline{\tau}}(n)$ is finite.

If σ is a spread, and X is a subset of σ_ω, then σ is a *subspread* of X iff $\overline{\overline{\sigma}} \subseteq \{\overline{\alpha}(n) \mid \alpha \in X,\, n \in \mathbb{N}\}$.

REMARK: a spread-law is nothing but the encoding of a tree as described in 0.0.0. The conditions on a spread-law σ ensure that we can construct an element of the corresponding spread step by step, in the course of time. For if we have chosen n values $\alpha(0), \alpha(1), \ldots, \alpha(n-1)$ of a sequence α in σ_ω such that $\sigma(\overline{\alpha}(i))=0$ for $i \leq n$, then by (ii) we can find at least one m in $\mathbb{N}$ such that choosing $\overline{\alpha}(n+1)=m$ gives $\sigma(\overline{\alpha}(n+1))=0$. A fan corresponds to the idea of a finitely branching tree.

DEFINITION: let σ be a spread, and let γ be an element of σ_ω. We say that γ is a *spread-function* from σ to $\mathbb{N}$ iff:

($\star$) $\forall \alpha \in \sigma\ \exists n \in \mathbb{N}\ [\,\gamma(\overline{\alpha}(n))>0 \wedge \forall m \in \mathbb{N}\, [m \neq n \rightarrow \gamma(\overline{\alpha}(m))=0]\,]$

If γ is a spread-function from σ to $\mathbb{N}$, α is in σ and $n \in \mathbb{N}$ is the unique natural number such that $\gamma(\overline{\alpha}(n))>0$, then we write $\gamma(\alpha)$ for $n-1$. If A is a subset of $\mathbb{N}$ then we say that γ is a spread-function from σ to A iff $\gamma(\alpha)$ is in A for all α in σ.

We say that γ is a *spread-function* from σ to σ_ω iff

($\star\star$) $\forall \alpha \in \sigma\ \forall n \in \mathbb{N}\ \exists m \in \mathbb{N}\ [\,\gamma(\overline{\alpha}(n+m))>0\,]$

If γ is a spread-function from σ to σ_ω, and α is in σ, then we inductively define an element $\gamma(\alpha)$ of σ_ω as follows. Put $\gamma(\alpha)(0)=\gamma(s_0)-1$ where $s_0=\mu t \in \mathbb{N}\, [\gamma(\overline{\alpha}(t))>0]$. Then for $n \in \mathbb{N}$ put $\gamma(\alpha)(n+1)=\gamma(s_{n+1})-1$ where $s_{n+1}=\mu t \in \mathbb{N}\, [t>s_n \wedge \gamma(\overline{\alpha}(t))>0]$. If ρ is a subspread of σ_ω then we say that γ is a spread-function from σ to ρ iff $\gamma(\alpha)$ is in ρ for all α in σ.

DEFINITION: if X and Y are subsets of σ_ω, then we write $X \times Y$ for the set $\{(x,y) \mid x \in X, y \in Y\}$ of ordered pairs of elements of X and Y respectively. We wish to see $X \times Y$ as a subset of σ_ω, and for this reason we code ordered pairs as follows. Let α and β be in σ_ω, then we write (α, β) for the unique γ in σ_ω such that $\gamma(n)=\lessdot \alpha(n), \beta(n) \gtrdot$ for all $n \in \mathbb{N}$. It is easy to see that if σ is a spread, then $\sigma \times \sigma$ is a spread as well. Similarly we define, for X, Y and Z subsets of σ_ω, a subset $X \times Y \times Z$ of σ_ω, and so on.

Let X and Y be subsets of σ_ω. We write $X \uplus Y$ for the subset $\{\lessdot 0 \gtrdot \star \alpha \mid \alpha \in X\} \cup \{\lessdot 1 \gtrdot \star \beta \mid \beta \in Y\}$ of σ_ω. If $(X_n)_{n \in \mathbb{N}}$ is a sequence of subsets of σ_ω, then we write $\biguplus_{n \in \mathbb{N}} X_n$ for the subset $\bigcup_{n \in \mathbb{N}} \{\lessdot n \gtrdot \star \alpha \mid \alpha \in X_n\}$ of σ_ω. When the context is clear we do not distinguish between the element β of $\biguplus_{n \in \mathbb{N}} X_n$ and the intended $\tilde{\beta}$ in $X_{\beta(0)}$ given by $\tilde{\beta}(n)=\beta(n+1)$. $X \uplus Y$ is called the *disjoint union* of X and Y, and $\biguplus_{n \in \mathbb{N}} X_n$ is called the *disjoint union* of $(X_n)_{n \in \mathbb{N}}$. The disjoint union of two spreads, or a sequence of spreads, is a spread.

DEFINITION: let σ be a spread, and a an element of $\overline{\overline{\sigma}}$. Define an element $\alpha_{a,\sigma}$ in σ as follows. Let $n \in \mathbb{N}$, then:

$$\alpha_{a,\sigma}(n) \underset{D}{=} \begin{cases} a_n & \text{if } n \leq lg(a) \\ \mu s \in \mathbb{N}\,[\,\sigma(\,\overline{\alpha}(n-1) \star \langle\!\langle s \rangle\!\rangle\,) = 0\,] & \text{else} \end{cases}$$

We mostly write α_a for $\alpha_{a,\sigma}$ when it is clear to which spread σ we refer. We use this definition to define a spread-function $\pi_{\omega,\sigma}$ from σ_ω to σ as follows. Let β be in σ_ω, and let $n \in \mathbb{N}$. Then:

$$\pi_{\omega,\sigma}(\beta)(n) \underset{D}{=} \begin{cases} \beta(n) & \text{if } \overline{\beta}(n+1) \in \overline{\overline{\sigma}} \\ \alpha_a(n) & \text{else, where } a = \max(\{\overline{\beta}(s) \mid s \leq n \wedge \overline{\beta}(s) \in \overline{\overline{\sigma}}\}) \end{cases}$$

Notice that $\pi_{\omega,\sigma}(\alpha) = \alpha$ for all α in σ. We leave it to the reader to define, for each fan τ, a spread-function $\pi_{2,\tau}$ from σ_2 to τ such that for all β in τ there is an α in σ_2 such that $\pi_{2,\tau}(\alpha) = \beta$.

0.0.4* DEFINITION: we define the *lexicographical ordering* $<_{\text{lex}}$ on $\overline{\overline{\sigma}}_\omega$ by putting: $a <_{\text{lex}} b$ iff $a \sqsubseteq b$, $a \neq b$ or $\exists i < \lg(a), lg(b)\ [\ \langle\!\langle a_0, \ldots a_{i-1} \rangle\!\rangle = \langle\!\langle b_0, \ldots b_{i-1} \rangle\!\rangle \wedge a_i < b_i\,]$.

LEMMA: there is a spread σ_{fan} such that each element α of σ_{fan} codes a subfan τ of σ_ω and for every subfan τ there is precisely one α in σ_{fan} coding τ.

PROOF: let τ be a subfan of σ_ω. We can code τ with an element α_τ of σ_ω as follows. For $n \in \mathbb{N}$ let $\{t_{n_0}, \ldots, t_{n_m}\}$ be the set $\overline{\overline{\tau}}(n)$ and such that $t_{n_i} <_{\text{lex}} t_{n_{i+1}}$ for $i < n_m$. Put $\alpha_\tau(n) = \langle\!\langle t_{n_0}, \ldots, t_{n_m} \rangle\!\rangle$. Clearly $\sigma_{\text{fan}} = \{\alpha_\tau \mid \tau \text{ is a fan}\}$ is a spread which satisfies the lemma •

0.0.5* DEFINITION: for $n \in \mathbb{N}$ we define the sequence $\underline{n}$ in σ_ω by putting: $\underline{n}(m) = n$ for all $m \in \mathbb{N}$. We define a spread $\sigma_{\mathbb{N}}$ as follows. For a in $\overline{\overline{\sigma}}_\omega$ put $\sigma_{\mathbb{N}}(a) = 0$ iff for all $i < lg(a)$: $a_i = a_0$. Then $\sigma_{\mathbb{N}}$ is the spread $\{\underline{n} \mid n \in \mathbb{N}\}$. We sometimes identify $\mathbb{N}$ with $\sigma_{\mathbb{N}}$.

Let $n \in \mathbb{N}$. Then we write σ_n for the subfan $\{\alpha \in \sigma_\omega \mid \forall m \in \mathbb{N}\,[\alpha(m) < n]\,\}$ of σ_ω. We write $\sigma_{n\text{mon}}$ for the subfan $\{\alpha \in \sigma_n \mid \forall m \in \mathbb{N}\,[\alpha(m) \leq \alpha(m+1)]\,\}$ of σ_n.

0.0.6* in the following subsections we state our axioms. However, we will not introduce a formal system, such as for instance in [Kleene&Vesley65]. Such a formalization is perhaps possible (although not within the formal system of [Kleene&Vesley65]), but if so it will probably make for difficult reading.

Our first axiom is the *principle of induction* for natural numbers, abbreviated with **Ind**, and stated thus:

$\boxed{\textbf{Ind}}$ let A be a subset of $\mathbb{N}$ such that $0 \in A$ and for all $n \in \mathbb{N}$: $n \in A$ implies $n+1 \in A$. Then $A = \mathbb{N}$, that is: $n \in A$ for all $n \in \mathbb{N}$.

The use of **Ind** other than in definitions is indicated by the words 'Basis' (where we prove that $0 \in A$) and 'Induction' (where we prove: $n \in A \rightarrow n+1 \in A$).

0.0.7 we now come to the intuitionistic foundations of this thesis. We postulate well-known axioms of choice, called $\mathbf{AC}_{00}$, $\mathbf{AC}_{01}$, $\mathbf{AC}_{10}$, $\mathbf{AC}_{11}$, $\mathbf{DC}_0$ and $\mathbf{DC}_1$. The first four can be found in [Kleene&Vesley65], [Gielen,deSwart&Veldman81], [Veldman85] and [Troelstra&vanDalen88] (where $\mathbf{AC}_{10}$ is called C-N and $\mathbf{AC}_{11}$ is called C-C). The last two are mentioned in [Troelstra&vanDalen88]. We also formulate the fan theorem **FT**, and present it as an axiom although it can be (and is) derived from the more fundamental axiom ^{x}26.3 in [Kleene&Vesley65] which is usually called the bar theorem. The fan theorem **FT** is not accepted in Bishop's school.

$\mathbf{AC}_{00}$ is a weak form of countable choice, which follows from the stronger version $\mathbf{AC}_{01}$. The latter still is far more limited than the axiom of countable choice in Bishop's school. The same holds, mutatis mutandis, for our axioms of dependent choice $\mathbf{DC}_0$ and $\mathbf{DC}_1$. $\mathbf{AC}_{10}$ is an axiom of continuous choice which follows from our general axiom of continuous choice $\mathbf{AC}_{11}$. These two axioms are not accepted in Bishop's school. Already a formally weaker form of $\mathbf{AC}_{10}$, called the continuity principle **CP**, is not accepted in Bishop's school.

For the sake of brevity we do not defend our axioms here (with the exception of $\mathbf{DC}_0$ and $\mathbf{DC}_1$). They are well-known, and the reader may ponder on them her- or himself. Else the reader may consult [Kleene&Vesley65], [Troelstra&vanDalen88] and especially [Gielen,deSwart&Veldman81] and [Veldman85].

We have marked many paragraphs with an asterisk *. This indicates that the paragraph is either straightaway acceptable in Bishop's school, or can easily be modified to become acceptable in Bishop's school. Mostly this means that we have used only $\mathbf{AC}_{00}$ and $\mathbf{AC}_{01}$ for the paragraph's results. The subject is developed in such a way that large portions carry an asterisk. We hope this will stimulate the interest of people in Bishop's school.

0.0.8* $\boxed{\mathbf{AC}_{00}}$ let A be a subset of $\mathbb{N}\times\mathbb{N}$ such that:

($\star$) $\forall n\in\mathbb{N}\ \exists m\in\mathbb{N}\ [\,(n,m)\in A\,]$

Then there is an α in σ_ω such that for each $n\in\mathbb{N}$: $(n,\alpha(n))$ is in A. We say that α *realizes* ($\star$).

$\boxed{\mathbf{AC}_{01}}$ let A be a subset of $\mathbb{N}\times\sigma_\omega$ such that:

($\star\star$) $\forall n\in\mathbb{N}\ \exists \alpha\in\sigma_\omega\ [\,(n,\alpha)\in A\,]$

Then there is an α in σ_ω such that for each $n\in\mathbb{N}$: $(n,\alpha_{[n]})$ is in A. We say that α *realizes* ($\star\star$).

0.0.9 $\boxed{\mathbf{AC}_{10}}$ let σ be a spread. Let A be a subset of $\sigma\times\mathbb{N}$ such that:

($\star$) $\forall \alpha\in\sigma\ \exists n\in\mathbb{N}\ [\,(\alpha,n)\in A\,]$

Then there is a spread-function γ from σ to $\mathbb{N}$ such that for each α in σ: $(\alpha,\gamma(\alpha))$ is in A. We say that γ *realizes* ($\star$).

$\boxed{\mathbf{AC}_{11}}$ let σ be a spread. Let A be a subset of $\sigma\times\sigma_\omega$ such that:

($\star\star$) $\forall \alpha\in\sigma\ \exists \beta\in\sigma_\omega\ [\,(\alpha,\beta)\in A\,]$

Then there is a spread-function γ from σ to σ_ω such that for each α in σ_ω: $(\alpha,\gamma(\alpha))$ is in A. We say that γ *realizes* ($\star\star$).

0.0.10* $\boxed{\mathbf{DC}_{0}}$ let $s\in\mathbb{N}$, and let A be a subset of $\mathbb{N}$. Suppose R is a subset of $A\times A$ such that:

($\star$) $s\in A\wedge\forall n\in A\ \exists m\in A\ [\,(n,m)\in R\,]$

Then there is an α in σ_ω such that $\alpha(0)=s$ and for each $n\in\mathbb{N}$: $(\alpha(n),\alpha(n+1))$ is in R.

$\boxed{\mathbf{DC}_{1}}$ let σ be a spread. Let δ be in σ, and let A be a subset of σ. Suppose R is a subset of $A\times A$ such that:

($\star\star$) $\delta\in A\wedge\forall \alpha\in A\ \exists \beta\in A\ [\,(\alpha,\beta)\in R\,]$

Then there is an γ in σ_ω such that $\gamma_{[0]}=\delta$ and for each $n\in\mathbb{N}$: $(\gamma_{[n]},\gamma_{[n+1]})$ is in R.

We justify $\mathbf{DC}_0$ thus. Since s is in A, we can safely begin the desired α by choosing: $\alpha(0)=s$. By ($\star$) above we can choose $\alpha(1)$ in $\mathbb{N}$ such that $(\alpha(0),\alpha(1))$ is in R. By ($\star$) above we can choose $\alpha(2)$ in $\mathbb{N}$ such that $(\alpha(1),\alpha(2))$ is in R, and so on...

Our justification of $\mathbf{DC}_1$ is similar, with a little more care. Since δ is in A, we can safely begin the desired γ by stipulating that $\gamma(0)=0$ and $\gamma(\lessdot 0 \gtrdot \star 0)=\delta(0)$, $\gamma(\lessdot 0 \gtrdot \star 1)=\delta(1)$, $\gamma(\lessdot 0 \gtrdot \star 2)=\delta(2)$, and so on... At the same time, while constructing δ and having begun the construction of γ, we begin the construction of a β in A such that (δ,β) is in R. That we can do so is guaranteed by ($\star\star$). We now fill in: $\gamma(\lessdot 1 \gtrdot \star 0)=\beta(0)$, $\gamma(\lessdot 1 \gtrdot \star 1)=\beta(1)$, $\gamma(\lessdot 1 \gtrdot \star 2)=\beta(2)$, and so on... At the same time, while constructing δ and β, we begin the construction of a β' in A such that (β,β') is in R. And so on...

0.0.11 we present the *continuity principle*, a weaker version of $\mathbf{AC}_{10}$.

$\boxed{\mathbf{CP}}$ let σ be a spread. Let A be a subset of $\sigma\times\mathbb{N}$ such that:

($\star$) $\forall\alpha\in\sigma\ \exists n\in\mathbb{N}\ [\,(\alpha,n)\in A\,]$

Then: $\forall\alpha\in\sigma\ \exists n\in\mathbb{N}\ \exists m\in\mathbb{N}\ \forall\beta\in\sigma\ [\,\overline{\alpha}(m)=\overline{\beta}(m)\rightarrow(\beta,n)\in A\,]$.

As an illustration we present a simple consequence of **CP**.

LEMMA: NOT $\forall\alpha\in\sigma_{2\text{mon}}\ [\,\alpha=\underline{0}\vee\exists n\in\mathbb{N}\,[\alpha(n)\neq 0]\,]$.

PROOF: suppose $\forall\alpha\in\sigma_{2\text{mon}}\ [\,\alpha=\underline{0}\vee\exists n\in\mathbb{N}\,[\alpha(n)\neq 0]\,]$. Then we have:

($\star$) $\forall\alpha\in\sigma_{2\text{mon}}\ \exists s\in\{0,1\}\ [\,(s=0\wedge\alpha=\underline{0})\vee(s=1\wedge\exists n\in\mathbb{N}\,[\alpha(n)\neq 0])\,]$

By **CP** applied to $\underline{0}$, there is an $s\in\{0,1\}$ and and an $m\in\mathbb{N}$ such that s realizes ($\star$) for all β in $\sigma_{2\text{mon}}\cap\overline{\underline{0}}(m)$. Suppose $s=1$. Then contradiction, since $\underline{0}$ is in $\sigma_{2\text{mon}}\cap\overline{\underline{0}}(m)$. Suppose $s=0$. Then contradiction, since the sequence β given by: $\overline{\beta}(m)=\overline{\underline{0}}(m)$ and $\beta(m+n)=1$ (for $n\in\mathbb{N}$) is in $\sigma_{2\text{mon}}\cap\overline{\underline{0}}(m)$. Contradiction •

0.0.12 finally we present our version of the famous *fan theorem*.

$\boxed{\mathbf{FT}}$ let τ be a fan. Let A be a subset of $\tau\times\mathbb{N}$ such that:

($\star$) $\forall\alpha\in\tau\ \exists n\in\mathbb{N}\ [\,(\alpha,n)\in A\,]$

Then there is an $N\in\mathbb{N}$ and a finite function h from $\overline{\overline{\tau}}(N)$ to $\mathbb{N}$ such that for all α in τ: $(\alpha,h(\overline{\alpha}(N)))$ is in A.

REMARK: notice that this implies that there is an $M\in\mathbb{N}$ such that for all α in τ: $\exists n\leq M\ [\,(\alpha,n)\in A\,]$. This latter, weaker formulation is the more usual one. But our version can then be derived from this weaker formulation and **CP**.

0.0.13* **AC**$_{00}$, **AC**$_{01}$, **AC**$_{10}$, **AC**$_{11}$, and **FT** correspond to *2.2, ^{x}2.1, *27.5, *27.4, and *27.8 in [Kleene&Vesley65]. The first four names given here are taken from [Gielen,deSwart&Veldman81].

The four axioms **Ind**, **AC**$_{11}$, **DC**$_1$ and **FT** suffice for this thesis. The weaker versions **AC**$_{00}$, **AC**$_{01}$, **AC**$_{10}$, **DC**$_0$ and **CP** are introduced since they occur in the literature, and also since they facilitate a translation to Bishop-style mathematics.

REMARK: if σ is a spread, and A is a *closed* subset of $\sigma\times\sigma_\omega$ (see def. 3.0.0) such that

($\star$) $\forall\alpha\in\sigma\ \exists\beta\in\sigma_\omega\ [\,(\alpha,\beta)\in A\,]$

Then the existence of a spread-function γ from σ to σ realizing ($\star$) is already a consequence of **DC**$_1$ combined with **AC**$_{10}$. In fact it suffices that for all α in σ the set $\{\beta\in\sigma_\omega \mid (\alpha,\beta)\in A\}$ is inhabited and closed. (If A is a spread, then the existence of a spread-function γ from σ to σ realizing ($\star$) is trivial, but observe that in general not every inhabited closed subset of $\sigma\times\sigma$ is a subspread.)

PROOF: (we prove our remark since we could not find it in the literature) let n be in $\mathbb{N}$. Let γ be a spread-function from σ to $\mathbb{N}$ such that for all α in σ: there is a β in σ_ω with the property that $\overline{\beta}(n)=\gamma(\alpha)$ and $(\alpha,\beta)\in A$. We then have:

($\star\star$) $\forall\alpha\in\sigma\ \exists m\in\mathbb{N}\ \exists\beta\in\sigma_\omega\ [\,m=\overline{\beta}(n+1)\wedge(\alpha,\beta)\in A\wedge\overline{\beta}(n)=\gamma(\alpha)\,]$

By **AC**$_{10}$ there is a spread-function δ from σ to $\mathbb{N}$ realizing ($\star\star$). Now let B be the subset of σ_ω given by:

$B=\{\gamma\in\sigma_\omega \mid \exists n\in\mathbb{N}\,[\gamma$ is a spread-function from σ to $\mathbb{N}$ such that for all α in σ: there is a β in σ_ω with the property that $\overline{\beta}(n)=\gamma(\alpha)$ and $(\alpha,\beta)\in A\,]\,\}$

Let R be the subset of $B\times B$ given by:

$R=\{(\gamma,\delta)\in B\times B \mid \exists n\in\mathbb{N}\,\forall\alpha\in\sigma\,[lg(\gamma(\alpha))=n=lg(\delta(\alpha))-1\wedge\gamma(\alpha)\sqsubset\delta(\alpha)]\,\}$

Let ϵ be the trivial spread-function from σ to $\mathbb{N}$ such that $\epsilon(\alpha)=0=\langle\!\langle\,\rangle\!\rangle$ for all α in σ. Then by our remark in the beginning of the proof we find:

($*$) $\epsilon\in B\wedge\forall\gamma\in B\ \exists\delta\in B\ [\,(\gamma,\delta)\in R\,]$

By **DC**$_1$ there is an η in σ_ω such that $\eta_{[0]}=\epsilon$ and for each $n\in\mathbb{N}$: $(\eta_{n]},\eta_{[n+1]})$ is in R. From η it is trivial to derive a spread-function γ from σ to σ_ω such that for all α in σ and all $n\in\mathbb{N}$: $\overline{\gamma(\alpha)}(n)=\eta_{[n]}(\alpha)$. Let α be arbitrary in σ. We find:

($**$) $\forall n\in\mathbb{N}\ \exists\beta\in\sigma_\omega\ [\,\overline{\beta}(n)=\overline{\gamma(\alpha)}(n)\wedge(\alpha,\beta)\in A\,]$

By $\mathbf{AC}_{01}$ there is a sequence $(\beta_n)_{n\in\mathbb{N}}$ in σ_ω realizing $(**)$. Clearly $\gamma(\alpha)=d_\omega$-$\lim(\beta_n)_{n\in\mathbb{N}}$. Since for all α the set $\{\beta\in\sigma_\omega \mid (\alpha,\beta)\in A\}$ is closed, we obtain that $(\alpha,\gamma(\alpha))$ is in A. Therefore γ realizes $(\star)$ $\bullet$

0.0.14* we will use the word *canonical* to indicate that a certain mathematical object can be found without the use of our axioms of choice. For example if σ is a spread, and a is in $\overline{\overline{\sigma}}$, then there is a canonical element of $\sigma\cap a$. We can take for instance α_a. We do not always specify the definition which shows that the object in question can be found canonically.

0.1 APARTNESS SPACES

0.1.0* DEFINITION: let X be a subset of σ_ω. An *apartness* on X is a subset $\#$ of $X\times X$ such that for all x, y, z in X:

(i) $x\#y$ implies $\exists n\in\mathbb{N}\,[\,x(n)\neq y(n)\,]$.

(ii) $x\#y$ iff $y\#x$.

(iii) $x\#y$ implies: $z\#x$ or $z\#y$.

An apartness $\#$ induces an equivalence relation $\equiv$ on X by putting: $x\equiv y$ iff NOT $x\#y$. We will always use this abbreviation. An *apartness space* is a pair $(X,\#)$ where X is a subset of σ_ω and $\#$ is an apartness on X. The *natural apartness* $\#_\omega$ on X is defined by putting, for x, y in X: $x\#_\omega y$ iff $\exists n\in\mathbb{N}\,[\,x(n)\neq y(n)\,]$.

REMARK: the constructive notion of apartness corresponds to an effective way to handle the classical notion of equivalence. Apartnesses therefore play a fundamental part in our account. Classically one often works with equivalence classes. We will refrain from doing so. Working with the sequences themselves, in the light of an apartness, seems more direct and natural. In the next paragraphs we explain how this can be done simply and effectively.

0.1.1* DEFINITION: let $(X,\#)$ be an apartness space, and let A be a subset of X. We say that A is a *subset* of $(X,\#)$ iff for all a in A and all x in X: $x\equiv a$ implies $x\in A$. We then also say that A is *closed under* $\equiv$*-equivalence*. A subset A of an apartness space

$(X, \#)$ is called *decidable* in $(X, \#)$ iff for all x: $x \in A$ or $x \notin A$. We write $X \backslash\backslash_A$ for the subset $\{x \in X \mid \forall a \in A\, [x \# a]\}$ of $(X, \#)$.

0.1.2* DEFINITION: let $(X, \#)$ and $(Y, \#_Y)$ be two apartness spaces. Then the *product* of $(X, \#)$ and $(Y, \#_Y)$ is the apartness space $(X \times Y, \#_{X \times Y})$, where the apartness $\#_{X \times Y}$ is given by: $(x, y) \#_{X \times Y} (w, z)$ iff $x \# w$ or $y \#_Y z$.

0.1.3* another fundamental notion is that of a function. Instead of taking it as primitive we follow the classical approach, for three reasons. The first is that the (non-primitive) notion of a spread-function beautifully captures the connotation of methodicity of 'function'. Spread-functions however are little known to people outside intuitionism, and ignored in Bishop's school. So we will be more easily understood if we use 'functions'. The second reason is that spread-functions are always defined on spreads, and this is a true limitation since certain spaces are not 'spreadlike' (see 1.1.5). The third reason is that the classical approach is perfectly adequate, since for us the word 'existence' has the same connotation of methodicity as the word 'function'.

DEFINITION: let $(X, \#)$ and $(Y, \#_Y)$ be two apartness spaces. A *weak function* from $(X, \#)$ to $(Y, \#_Y)$ is a subset f of $(X \times Y, \#_{X \times Y})$ such that:

(i) $\forall x \in X\ \exists y \in Y\ [(x, y) \in f]$.

(ii) $\forall x \in X\ \forall y, z \in Y\ [((x, y) \in f \wedge (x, z) \in f) \rightarrow y \equiv_Y z]$.

When the context is clear we abbreviate (i) and (ii) with: $\forall x \in X\ \exists^{!\equiv} y \in Y\ [(x, y) \in f]$. A subset f of $(X \times Y, \#_{X \times Y})$ is called a *function* from $(X, \#)$ to $(Y, \#_Y)$ iff in addition to (i) and (ii) we have:

(iii) $\forall x, w \in X\ \forall y, z \in Y\ [((x, y) \in f \wedge (w, z) \in f \wedge y \#_Y z) \rightarrow x \# w]$.

Let f be a weak function from $(X, \#)$ to $(Y, \#_Y)$. Then for x in X we write $f(x)$ for the subset $\{y \in Y \mid (x, y) \in f\}$ of Y. For y in Y such that $(x, y) \in f$ we write: $f(x) \equiv_Y y$. For Z in X we write $f(x) \#_Y f(z)$ to abbreviate: $\exists y \in f(x)\ \exists w \in f(z)\ [y \#_Y w]$. Similar abbreviations using '$f(x)$' are left to the understanding of the reader. In addition let g be a weak function from $(Y, \#_Y)$ to $(Z, \#_Z)$, a third apartness space. Then we write $g \circ f$ for the subset $\{(x, z) \in X \times Z \mid \exists y \in Y\, [f(x) \equiv_Y y \wedge g(y) \equiv_Z z\}$ of $X \times Z$, which is a weak function from $(X, \#)$ to $(Z, \#_Z)$. We say that $g \circ f$ is the *composition* of f and g.

0.1.4* DEFINITION: a function f from $(X, \#)$ to $(Y, \#_Y)$ is called *injective* (an *injection*) iff for all x, z in X: $x \# z$ implies $f(x) \#_Y f(z)$. We say that f is *surjective* (a *surjection*) iff for all y in Y there is an x in X such that $f(x) \equiv_Y y$. Finally, f is *bijective* (a *bijection*) iff f is both injective and surjective. Notice that in this case the subset $f^{-1} = \{(y,x) \mid (x,y) \in f\}$ is a bijection from $(Y, \#_Y)$ to $(X, \#)$ and $f^{-1} \circ f(x) \equiv x$ for all x in X.

0.1.5 an important intuitionistic result is the following lemma, the proof of which requires $\mathbf{AC}_{11}$. It says that a weak function from an apartness spread to an arbitrary apartness space can be represented by a spread-function. We often use it to reduce functions to elements of σ_ω, in order to move them into the scope of our axioms $\mathbf{AC}_{01}$, $\mathbf{DC}_1$ and $\mathbf{AC}_{11}$.

LEMMA: let f be a weak function from an apartness spread $(\sigma, \#)$ to an arbitrary apartness space $(Y, \#_Y)$. Then there is a spread-function γ from σ to Y such that for all α in σ: $(\alpha, \gamma(\alpha))$ is in f.

PROOF: it suffices to remember that Y is a subset of σ_ω, by definition of 'apartness space'. We find:

$(\star)$ $\forall \alpha \in \sigma \; \exists \beta \in \sigma_\omega \; [(\alpha, \beta) \in f]$

By $\mathbf{AC}_{11}$ there is a spread-function γ from σ to σ_ω realizing $(\star)$ •

0.2 THE REAL NUMBERS

0.2.0* obviously the real numbers are fundamental to our account. We will build them as the members of a spread, called $\mathbb{R}$ as usual. Many a construction is possible, each having its own advantages and disadvantages. We assume familiarity with the real numbers, and choose and discuss just one such construction. In essence we identify a real number with its ternary expansion, where we have to allow for the extra digit 3 since we must be able to 'jump back'.

DEFINITION: we define: $\mathbb{R} \underset{D}{=} \mathbb{N} \times \sigma_4$. We define a subset $\approx_{\mathbb{R}}$ of $\overline{\overline{\mathbb{R}}} \times \overline{\overline{\mathbb{R}}}$ as follows. Let a in $\overline{\overline{\mathbb{R}}}$. Put $m_a = -1^{a_0} \cdot [\frac{a_0}{2}]$ (then $m_a \in \mathbb{Z}$) and:

$$a_{\mathbb{R}} = m_a + \textstyle\sum_{0<i<lg(a), a_i<3} a_i \cdot 3^{-i} - \sum_{0<i<lg(a), a_i=3} a_i \cdot 3^{-i-1}.$$

Let a, b be in $\overline{\overline{\mathbb{R}}} \times \overline{\overline{\mathbb{R}}}$. If $lg(a)=0$ or $lg(b)=0$ then put $a \approx_{\mathbb{R}} b$ and $b \approx_{\mathbb{R}} a$. Else put $a \approx_{\mathbb{R}} b$

iff: $a_{\mathbb{R}} - \frac{1}{2}3^{-lg(a)+1} \leq b_{\mathbb{R}} + 3^{-lg(b)+1} \leq a_{\mathbb{R}} + 3^{-lg(a)+1}$ or $b_{\mathbb{R}} - \frac{1}{2}3^{-lg(b)+1} \leq a_{\mathbb{R}} + 3^{-lg(a)+1} \leq b_{\mathbb{R}} + 3^{-lg(b)+1}$ Write $\not\approx_{\mathbb{R}}$ for the complement of $\approx_{\mathbb{R}}$ in $\overline{\overline{\mathbb{R}}}$. The standard apartness $\#_{\mathbb{R}}$ on $\mathbb{R}$ is defined by putting $\alpha \#_{\mathbb{R}} \beta$ iff there is an $n \in \mathbb{N}$ such that $\overline{\alpha}(n) \not\approx_{\mathbb{R}} \overline{\beta}(n)$.

An α in $\mathbb{R}$ is called a *ternary real number* iff for all $n \in \mathbb{N}$: $\alpha(n+1) < 3$. The set of the ternary real numbers will be denoted $\mathbb{R}_3$. We also define the *unit interval* $[0,1]$ by: $[0,1] \underset{D}{=} \{\alpha \in \mathbb{R} \mid \forall n \in \mathbb{N}\, [\, 0 \leq \overline{\alpha}(n+1)_{\mathbb{R}} \leq 1\,]\,\}$. Finally put $[0,1]_3 \underset{D}{=} [0,1] \cap \mathbb{R}_3$.

0.2.1* DEFINITION: we rely on the reader's familiarity with the real numbers, and leave it to her or him to define the addition $+_{\mathbb{R}}$ and multiplication $\cdot_{\mathbb{R}}$ on $\mathbb{R}$ as an exercise in ternary arithmetic. A similar exercise is to define the *absolute value* function $|\ |_{\mathbb{R}}$. We define: $\alpha <_{\mathbb{R}} \beta$ iff there is an $n \in \mathbb{N}$ such that: $\overline{\alpha}(n) \not\approx_{\mathbb{R}} \overline{\beta}(n)$ and $\overline{\alpha}(n)_{\mathbb{R}} < \overline{\beta}(n)_{\mathbb{R}}$. Also $\alpha >_{\mathbb{R}} \beta$ iff $\beta <_{\mathbb{R}} \alpha$. Finally $\alpha \leq_{\mathbb{R}} \beta$ iff NOT $\alpha >_{\mathbb{R}} \beta$, and $\alpha \geq_{\mathbb{R}} \beta$ iff NOT $\alpha <_{\mathbb{R}} \beta$. In practice we mostly omit the subscripts. Let α, β be in $\mathbb{R}$ such that $\alpha \leq \beta$. We write $[\alpha, \beta]$ for the subset $\{\gamma \in \mathbb{R} \mid \alpha \leq \gamma \leq \beta\}$, and $[\alpha, \beta]_3$ for $[\alpha, \beta] \cap \mathbb{R}_3$.

We consider $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ to be subsets of $\mathbb{R}$. For example we simply write 0 for the sequence $\underline{0}$ in $\mathbb{R}$, and 1 for the sequence $\langle\!\langle 2 \rangle\!\rangle \star \underline{0}$ in $\mathbb{R}$.

REMARK: one of the consequences of **CP** is: NOT every real number is equivalent to a ternary real number. This was already noted by Brouwer, see [Brouwer22]. For suppose every real number is equivalent to a ternary real number. Then for every real number α we can decide: $\alpha \leq 0$ or $\alpha \geq 0$. So we then have:

$(\star)$ $\forall \alpha \in \mathbb{R}\ \exists s \in \{0,1\}\ [\,(s=0 \wedge \alpha \leq 0) \vee (s=1 \wedge \alpha \geq 0)\,]$

By **CP** applied to $\underline{0}$ we see that there is an $m \in \mathbb{N}$ such that: $\forall \beta \in \mathbb{R}\ [\,\overline{\beta}(m) = \overline{\underline{0}}(m) \rightarrow \beta \leq 0\,]$ or $\forall \beta \in \mathbb{R}\ [\,\overline{\beta}(m) = \overline{\underline{0}}(m) \rightarrow \beta \geq 0\,]$. Contradiction.

0.2.2 another well-known consequence of **CP** is the following. Let A be a decidable subset of $([0,1], \#_{\mathbb{R}})$, meaning that for all α in $\mathbb{R}$: $\alpha \in A$ or $\alpha \notin A$. Then $A = \emptyset$ or $A = [0,1]$. We say: the *continuum* is *unsplittable*.

0.2.3* LEMMA: let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$. Let γ and δ be in $\mathbb{R}$ such that $\gamma < \delta$. Then there is an α in $\mathbb{R}$ such that: $\gamma < \alpha < \delta$ and for all $n \in \mathbb{N}$: $\alpha \#_{\mathbb{R}} \alpha_n$.

DEFINITION: let A be a subset of $(\mathbb{R}, d_{\mathbb{R}})$, and let α be in $\mathbb{R}$. We say that α is the *supremum* of A iff $\forall \gamma \in A\ [\,\gamma \leq \alpha\,]$ and for all β in $\mathbb{R}$: $\forall \gamma \in A\ [\,\gamma \leq \beta\,]$ implies $\alpha \leq \beta$. We then write $\sup(A)$ for α. Similarly we say that α is the *infimum* of A iff $\forall \gamma \in A\ [\,\gamma \geq \alpha\,]$ and for all β in $\mathbb{R}$: $\forall \gamma \in A\ [\,\gamma \geq \beta\,]$ implies $\alpha \geq \beta$. We then write $\inf(A)$ for α. We say that $\sup(A)$ $(\inf(A))$ *exists* iff there is an α in $\mathbb{R}$ such that α is the supremum (infimum) of A.

0.2.4* we assume the reader is familiar with the real number π given by: $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}\ldots$. It can be shown that π is positively irrational, that is: $\pi \# q$ for all q in $\mathbb{Q}$. From this it follows that there is a unique ternary real number α_π such that $\alpha_\pi \equiv \pi$. We call α_π the *ternary expansion* of π. It also follows that π has a (unique) *decimal* expansion with digits in $\{0,\ldots,9\}$. Notice that this unique decimal expansion can be seen as an element of σ_{10}. Brouwer often used the decimal expansion of π in the following way. If certain classical statements were true intuitionistically, then we would obtain all sorts of hitherto unknown information about the decimal expansion of π, answering questions such as: is there a sequence of ninety-nine consecutive nines in the decimal expansion of π? By a *Brouwerian counterexample* to a statement P, we mean an example such that if P were true for the example, then we can answer (a question very similar to) the question above. Let us make this precise.

DEFINITION: let $n\in\mathbb{N}$. We write $n=k_{99}$ iff n is the smallest natural number s such that for all $i<99$ in $\mathbb{N}$: $\alpha_\pi(s+i)=2$. We write $n<k_{99}$ iff for all $m\leq n$ in $\mathbb{N}$: NOT $m=k_{99}$. We define an element $\alpha_{k_{99}}$ in $\sigma_{2\text{mon}}$ as follows: $\alpha_{k_{99}}(n)=0$ iff $n<k_{99}$ and $\alpha_{k_{99}}(n)=1$ iff there is an $m\in\mathbb{N}$ such that $n-m=k_{99}$.

We believe that no one as of yet has a method to find out whether $\exists n\in\mathbb{N}\ [n=k_{99}]$ or $\forall n\in\mathbb{N}\ [n<k_{99}]$. Therefore we call statements which implicitly answer our question *daring*. The use of Brouwerian counterexamples has become widespread both in intuitionism and in Bishop's school.

0.3 METRIC SPACES

0.3.0* DEFINITION: let X be a subset of σ_ω. A *metric* on X is a function from $(X\times X, \#_\omega)$ to $\mathbb{R}_{\geq 0}$ such that for all x, y, z in X:

(i) $d(x,y)>0$ implies $x\#_\omega y$.

(ii) $d(x,y)\equiv d(y,x)$.

(iii) $d(x,z)\leq d(x,y)+d(y,z)$.

Property (iii) of a metric d is often called the *triangle inequality*. A *metric space* is a pair (X,d) where X is a subset of σ_ω and d is a metric on X. A metric d on X induces an apartness $\#_d$ on X, defined by: $x\#_d y$ iff $d(x,y)>0$. A *subset* of (X,d) is a subset of $(X,\#_d)$ (in the sense of 0.1.1). A metric space (X,d) is called *separable* iff there is a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that for all y in X and all $m\in\mathbb{N}$ there is an $n\in\mathbb{N}$ such that $d(y,x_n)<2^{-m}$. We then say that $(x_n)_{n\in\mathbb{N}}$ is *dense* in (X,d).

Finally let x be an element of the metric space (X,d), and let α be in $\mathbb{R}_{\geq 0}$. We write $B(x,\alpha)$ for the subset $\{y\in X \mid d(x,y)<\alpha\}$ of (X,d), and we write $cB(x,\alpha)$ for the subset $\{y\in X \mid d(x,y)\leq\alpha\}$ of (X,d).

REMARK: we shall mostly be concerned with separable metric spaces, see also convention 1.0.7. Our notion of a 'metric' parallels the classical notion of a 'pseudo-metric'.

0.3.1* DEFINITION: we define a metric d_ω on σ_ω by putting, for α and β in σ_ω: $d_\omega(\alpha,\beta) \underset{D}{=} \inf(\{2^{-n} \mid n\in\mathbb{N} \wedge \overline{\alpha}(n)=\overline{\beta}(n)\})$. We define a metric $d_{\mathbb{R}}$ on $\mathbb{R}$ by putting: $d_{\mathbb{R}}(\alpha,\beta) \underset{D}{=} |\alpha-\beta|$, for α and β in $\mathbb{R}$.

0.3.2* DEFINITION: let (X,d) and (Y,d_Y) be two metric spaces. The *product* of (X,d) and (Y,d_Y) is the metric space $(X\times Y, d_{X\times Y})$, where $d_{X\times Y}$ is the metric given by: $d_{X\times Y}((x,w),(y,z)) \underset{D}{=} \sup(d(x,y), d_Y(w,z))$. A subset f of $(X\times Y, d_{X\times Y})$ is a *(weak) function* from (X,d) to (Y,d_Y) iff f is a (weak) function from $(X,\#_d)$ to $(Y,\#_{d_Y})$.

0.3.3* DEFINITION: let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X,d). Then $(x_n)_{n\in\mathbb{N}}$ is a *Cauchy-sequence* in (X,d) iff:

$\forall n\in\mathbb{N}\ \exists N\in\mathbb{N}\ \forall m\in\mathbb{N}\ [\,d(x_N,x_{N+m})<2^{-n}\,]$.

Let x in X. We say that $(x_n)_{n\in\mathbb{N}}$ *d-converges* to x iff:

$\forall n\in\mathbb{N}\ \exists N\in\mathbb{N}\ \forall m\in\mathbb{N}\ [\,d(x,x_{N+m})<2^{-n}\,]$.

We also say that $(x_n)_{n\in\mathbb{N}}$ *converges* in (X,d) (to x). We call (X,d) *complete* iff each Cauchy-sequence in (X,d) converges in (X,d).

0.3.4* DEFINITION: let (X,d) be a metric space. Write $X^{\mathbb{N}}$ for the subset $\{\alpha\in\sigma_\omega \mid \forall n\in\mathbb{N}\,[\,\alpha_{[n]}\in X\,]\}$ of σ_ω. Let A be a subset of (X,d). Put

$\overline{A} \underset{D}{=} \{\alpha\in X^{\mathbb{N}} \mid (\alpha_{[n]})_{n\in\mathbb{N}}$ is a Cauchy-sequence in (X,d) $\wedge\ \forall n\in\mathbb{N}\,[\alpha_{[n]}\in A]\}$.

Suppose (X,d) is complete. Then we can define a function d-lim from $(\overline{X},d_\omega)$ to (X,d) by putting:

$d\text{-lim} \underset{D}{=} \{(\alpha,x)\in\overline{X}\times X \mid (\alpha_{[n]})_{n\in\mathbb{N}}$ d-converges to $x\}$.

We define a metric $\overline{d}$ on $\overline{X}$ by putting $\overline{d}(\alpha,\beta) \underset{D}{=} d_{\mathbb{R}}\text{-lim}(d(\alpha_{[n]},\beta_{[n]})_{n\in\mathbb{N}}$, for α and β in $\overline{X}$. The metric space $(\overline{X},\overline{d})$ is called the *completion* of (X,d) . We mostly write $\overline{(X,d)}$ for $(\overline{X},\overline{d})$, and simply d again for $\overline{d}$.

THEOREM: if (X,d) is a metric space, then the completion $\overline{(X,d)}$ is a complete metric space.

DEFINITION: we define the *Hilbert cube* $(\mathcal{Q},d_{\mathcal{Q}})$ as follows: $\mathcal{Q} \underset{D}{=} [-1,1]^{\mathbb{N}}$ and $d_{\mathcal{Q}}$ is the metric on $\mathcal{Q}$ defined by putting $d_{\mathcal{Q}}(\alpha,\beta) \underset{D}{=} \sum_{n\in\mathbb{N}} 2^{-n}\cdot|\alpha_{[n]}-\beta_{[n]}|$, for α , β in $\mathcal{Q}$.

0.3.5* DEFINITION: a metric space (X,d) is called *precompact* iff there is a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

$\forall n\in\mathbb{N}\ \exists m\in\mathbb{N}\ \forall x\in X\ [\exists i\leq m\,[d(x,x_i)<2^{-n}]]$

0.3.6* DEFINITION: let d and d_1 be two metrics on a subset X of σ_ω . We say that d_1 is *d-equivalent* iff for all x in X and all α in $\mathbb{R}^+$ there are β and γ in $\mathbb{R}^+$ such that $B_d(x,\gamma)\subseteq B_{d_1}(x,\alpha)$ and $B_{d_1}(x,\beta)\subseteq B_d(x,\alpha)$. We call d_1 *strongly d-equivalent* iff in addition each Cauchy-sequence in (X,d) is a Cauchy-sequence in (X,d_1) and vice versa.

0.4 CONTINUOUS FUNCTIONS

0.4.0* DEFINITION: let (X,d) and (Y,d_Y) be two metric spaces. A function f from (X,d) to (Y,d_Y) is *(d,d_Y)-continuous* iff:

$\forall x\in X\ \forall n\in\mathbb{N}\ \exists m\in\mathbb{N}\ \forall y\in X\ [d(x,y)<2^{-m}\rightarrow d_Y(f(x),f(y))<2^{-n}]$.

When the context is clear we simply say that f is continuous. We say that f is *uniformly continuous* iff:

$\forall n\in\mathbb{N}\ \exists m\in\mathbb{N}\ \forall x,y\in X\ [d(x,y)<2^{-m}\rightarrow d_Y(f(x),f(y))<2^{-n}]$.

0.4.1 A remarkable consequence of **CP** is that every weak function from $(\mathbb{R}, d_{\mathbb{R}})$ to a separable metric space (X, d) is continuous. We give a simple proof.

LEMMA: let f be a weak function from $(\mathbb{R}, d_{\mathbb{R}})$ to a separable metric space (X, d). Then: $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N}\, \forall \beta \in \mathbb{R}\, [\, d_{\mathbb{R}}(0, \beta) < 2^{-m} \to d(f(0), f(\beta)) < 2^{-n}\,]$.

PROOF: let $(x_n)_{n \in \mathbb{N}}$ be dense in (X, d). Let $n \in \mathbb{N}$. We have:

$(\star)$ $\forall \alpha \in \mathbb{R}\, \exists s \in \mathbb{N}\, [\, d(x_s, f(\alpha)) < 2^{-n-1}\,]$

By **CP** applied to $\underline{0}$ we find a $p \in \mathbb{N}$ and an $s \in \mathbb{N}$ such that for all γ in $\mathbb{R}$: $\overline{\gamma}(p) = \overline{\underline{0}}(p)$ implies $d(f(\gamma), x_s) < 2^{-n-1}$. But for all β in $\mathbb{R}$: $d_{\mathbb{R}}(\beta, 0) < 2^{-2p}$ implies that there is a γ in $\mathbb{R}$ such that: $\overline{\gamma}(p) = \overline{\underline{0}}(p)$ and $\gamma \equiv_{\mathbb{R}} \beta$. Taking $m = 2p$, the lemma now follows from the fact that f is a weak function •

THEOREM: let f be a weak function from $(\mathbb{R}, d_{\mathbb{R}})$ to a separable metric space (X, d). Then f is continuous.

PROOF: let α be in $\mathbb{R}$. We must show:

$(\star)$ $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N}\, \forall \beta \in \mathbb{R}\, [\, d_{\mathbb{R}}(\alpha, \beta) < 2^{-m} \to d(f(\alpha), f(\beta)) < 2^{-n}\,]$

Let $f_{+\alpha}$ be the function from $\mathbb{R}$ to $\mathbb{R}$ given by $f_{+\alpha}(\beta) = \beta + \alpha$. Now $(\star)$ follows from the lemma above applied to the weak function $f \circ f_{+\alpha}$ •

REMARK: in this proof we use the linear structure of $\mathbb{R}$, see section 0.5. We will present a 'deeper' reason for the continuity of a weak function from $(\mathbb{R}, d_{\mathbb{R}})$ to a separable metric space (X, d) in the course of the next chapters, see theorems 1.1.0, 1.2.4, 3.3.10 and 3.3.12.

0.4.2* LEMMA: let f be a uniformly continuous function from a metric space (X, d) to another metric space (Y, d_Y). Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy-sequence in (X, d). Then $(f(x_n))_{n \in \mathbb{N}}$ is a Cauchy-sequence in (Y, d_Y).

COROLLARY: there is a uniformly continuous function $\tilde{f}$ from $\overline{(X, d)}$ to $\overline{(Y, d_Y)}$ such that for all x in X: $\tilde{f}(x) \equiv_Y f(x)$.

0.4.3* THEOREM: let (X, d) be a precompact metric space. Let f be a uniformly continuous function from (X, d) to $(\mathbb{R}, d_{\mathbb{R}})$. Then $\sup(\{f(x) \mid x \in X\})$ and $\inf(\{f(x) \mid x \in X\})$ exist.

0.4.4* LEMMA: let d be a metric on a fan τ such that d is a uniformly continuous function from $(\tau\times\tau, d_\omega)$ to $\mathbb{R}$. Then (τ, d) is precompact.

0.4.5* DEFINITION: let (X,d) and (Y,d_Y) be metric spaces. We say that (X,d) *coincides isometrically* with (Y,d_Y) iff there is a bijective function f from (X,d) to (Y,d_Y) such that for all x, y in X: $d_Y(f(x),f(y))=d(x,y)$.

LEMMA: let (X,d) be a complete and precompact metric space. Then (X,d) coincides isometrically with a metric fan (τ, d_τ).

0.4.6 THEOREM: a continuous function f from a metric fan (τ,d) to a metric space (Y,d_Y) is uniformly continuous.

PROOF: this is a straightforward consequence of the fan theorem **FT** •

COROLLARY:

(i) every metric fan is precompact, see lemma 0.4.4.

(ii) $\sup(\{f(\alpha)\,|\,\alpha\in\tau\})$ and $\inf(\{f(\alpha)\,|\,\alpha\in\tau\})$ exist.

0.4.7* LEMMA: (compare [Bishop&Bridges85, thm.4.4.9]) let f be a uniformly continuous function from a precompact metric fan (τ,d) to $\mathbb{R}$. Then there is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, such that for all α in $\mathbb{R}$ for which $\forall n\in\mathbb{N}[\alpha \#_{\mathbb{R}} \alpha_n]$: $\{\beta\in\tau \mid f(\beta)\leq\alpha\}$ is empty, or $\{\beta\in\tau \mid f(\beta)\leq\alpha\}$ is a subfan of τ.

PROOF: let $(a_n)_{n\in\mathbb{N}}$ be an enumeration of $\overline{\overline{\tau}}$. For $n\in\mathbb{N}$ put $\alpha_n=\inf(\{f(\beta) \mid \beta\in\tau\cap a_n\}$ which exists by theorem 0.4.3. It is easy to see that $(\alpha_n)_{n\in\mathbb{N}}$ is as required •

COROLLARY: let f be a continuous function from a metric fan (τ,d) to $\mathbb{R}$. Then there is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that for all α in $\mathbb{R}$ for which $\forall n\in\mathbb{N}[\alpha \#_{\mathbb{R}} \alpha_n]$: $\{\beta\in\tau \mid f(\beta)\geq\alpha\}$ is empty, or $\{\beta\in\tau \mid f(\beta)\geq\alpha\}$ is a subfan of τ.

PROOF: apply the lemma to the function $-f$ •

COROLLARY: let f be a continuous function from a metric fan (τ,d) to $\mathbb{R}$. Suppose γ and δ are in $\mathbb{R}$ such that $\gamma<\delta$ and $\gamma\in f(\tau)$. Then there is an α in $\mathbb{R}$ such that: $\gamma<\alpha<\beta$ and $\{\beta\in\tau \mid f(\beta)\leq\alpha\}$ is a subfan of τ.

PROOF: let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence as in the conclusion of the lemma above. By lemma 0.2.3 we can construct an α in $\mathbb{R}$ such that $\gamma<\alpha<\beta$ and for all $n\in\mathbb{N}$: $\alpha \#_{\mathbb{R}} \alpha_n$. Since $\{\beta\in\tau \mid f(\beta)\leq\alpha\}$ is inhabited, it must be a subfan of τ •

0.4.8* THEOREM: let (X,d) and (Y,d_Y) be two metric spaces. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of (uniformly) continuous functions from (X,d) to (Y,d_Y) such that for all $n\in\mathbb{N}$ and for all x in X: $d_Y(f_n(x),f_{n+1}(x))<2^{-n}$. Then the function f from (X,d) to $\overline{(Y,d_Y)}$ defined by $f(x)=d_Y\text{-}\lim(f_n(x))_{n\in\mathbb{N}}$ is (uniformly) continuous.

0.5 LINEAR SPACES

0.5.0* DEFINITION: a *linear space* is a quadruple $<(L,d_L), +, \cdot, \mathbf{0}>$ where (L,d_L) is a *separable* metric space, $+$ is a continuous function from $(L\times L, d_{L\times L})$ to (L,d_L), $\cdot$ is a continuous function from $(\mathbb{R}\times L, d_{\mathbb{R}\times L})$ to (L,d_L), and $\mathbf{0}$ is an element of L such that for all x, y, z, in L and all α, β in $\mathbb{R}$:

(i) $\alpha\cdot(x+y)\equiv(\alpha\cdot y+\alpha\cdot x)$.

(ii) $x+\mathbf{0}\equiv x$ and $x+(-1\cdot x)\equiv\mathbf{0}$.

(iii) $(x+y)+z\equiv x+(y+z)$

(iv) $\alpha\cdot\mathbf{0}\equiv\mathbf{0}$ and $0\cdot x\equiv\mathbf{0}$.

(v) $(\alpha+\beta)\cdot x\equiv\alpha\cdot x+\beta\cdot x$ and $\alpha\cdot(\beta\cdot x)\equiv(\alpha\beta)\cdot x$.

We mostly write: 'let (L,d_L) be a linear space' as abbreviation for 'let $<(L,d_L), +, \cdot, \mathbf{0}>$ be a linear space'. Also we frequently write αx for $\alpha\cdot x$.

0.5.1* DEFINITION: a function $\| \ \|$ from a linear space (L,d_L) to $(\mathbb{R}_{\geq 0}, d_{\mathbb{R}})$ is called a *norm* on (L,d_L) iff for all x, y in L and all α, β in $\mathbb{R}$:

(i) $\|x\|>0$ iff $x\#\mathbf{0}$.

(ii) $\|x+y\|\leq\|x\|+\|y\|$.

(iii) $\|\alpha x\|\#|\beta|\cdot\|x\|$ iff: $x\#\mathbf{0}$ and $|\alpha|\#_{\mathbb{R}}|\beta|$.

A norm $\| \ \|$ on a linear space (L,d_L) induces a metric $d_{\| \ \|}$ on L by putting: $d_{\| \ \|}(x,y) \underset{D}{=} \|x-y\|$ for x, y in L. A *normed linear space* is a quintuple $<(L,d_L), +, \cdot, \mathbf{0}, \| \ \|>$

such that $\| \ \|$ is a norm on the linear space $<(L,d_L)\,,+\,,\cdot\,,\ \mathbf{0}\ >$, and such that in addition $d_L(x,y)\equiv d_{\| \ \|}(x,y)$ for all $x\,,y$ in L. We mostly write: 'let $(L,\| \ \|)$ be a normed linear space' as abbreviation for 'let $<(L,d_L)\,,+\,,\cdot\,,\ \mathbf{0}\ ,\| \ \|>$ be a normed linear space'. A *Banach space* is a normed linear space $(L,\| \ \|)$ such that $(L,d_{\| \ \|})$ is a complete metric space.

0.5.2* DEFINITION: let (L,d_L) be a linear space. If A is a subset of (L,d_L), then we define the *convex hull* of A, notation $conv(A)$, as follows:

$$conv(A) \underset{D}{=} \bigcup_{n\in\mathbb{N}}\{\sum_{i\leq n}\rho_i\cdot x_i \mid \rho_0,\ldots\rho_n\in\mathbb{R}_{\geq 0}\wedge \sum_{i\leq n}\rho_i\equiv_{\mathbb{R}}1 \mid x_0,\ldots,x_n\in A\}$$

A subset A of (L,d_L) is called *convex* iff $A=conv(A)$. A linear space (L,d_L) is called *locally convex* iff for all x in L and all $n\in\mathbb{N}$: there is an $m\in\mathbb{N}$ such that $conv(B(x,2^{-m}))\subseteq B(x,2^{-n})$.

REMARK: every normed linear space is locally convex.

0.5.3* DEFINITION: let (τ,d) be a metric fan, and let (Y,d_Y) be a metric space. We write $C((\tau,d),(Y,d_Y))$ for the set of of all uniformly continuous spread-functions from (τ,d) to (Y,d_Y). We define a metric $d_{\sup}$ on $C((\tau,d),(Y,d_Y))$ by putting:

$$d_{\sup}(f,g) \underset{D}{=} \sup(\{d_Y(f(\alpha),g(\alpha)) \mid \alpha\in\tau\})$$

which exists by lemma 0.4.3 combined with lemma 0.4.4. By theorem 0.5.6 we have that if (Y,d_Y) is a locally convex linear space (L,d_L), then $(C((\tau,d),(L,d_L)),d_{\sup})$ is a linear space if we define $f+g$, $\alpha\cdot f$ and $\mathbf{0}_{C((\tau,d),(L,d_L))}$ in the obvious way, for $f\,,g$ in $C((\tau,d),(L,d_L))$ and α in $\mathbb{R}$. If (Y,d_Y) is a normed linear space $(L,\| \ \|)$, then we define a norm $\| \ \|_{\sup}$ on $(C((\tau,d),(L,\| \ \|)),d_{\sup})$ by putting: $\|f\|_{\sup} \underset{D}{=} d_{\sup}(f,\ \mathbf{0}_{C((\tau,d),(L,\| \ \|))})$.

0.5.4* LEMMA: let (τ,d) be a metric fan, and let (Y,d_Y) be a complete metric space. Then $(C((\tau,d),(Y,d_Y)),d_{\sup})$ is a complete metric space.

PROOF: by lemma 0.4.8 a Cauchy-sequence in $(C((\tau,d),(Y,d_Y)),d_{\sup})$ converges to a uniformly continuous function from (τ,d) to (Y,d_Y). By lemma 0.1.5 such a function can be represented by a spread-function. Therefore every Cauchy-sequence in $(C((\tau,d),(Y,d_Y)),d_{\sup})$ converges in $(C((\tau,d),(Y,d_Y)),d_{\sup})$ •

0.5.5 in classical mathematics one can prove that if (X,d) is a compact metric space and (Y,d_Y) is another metric space, then $(C(X,Y),d_{\sup})$ is separable (see for instance [vanMill89, prp.1.3.3.]). The next proposition shows that the intuitionistic situation is different, even if (Y,d_Y) is a metric fan.

PROPOSITION: NOT for every metric fan (τ,d): $(C(([0,1],d_{\mathbb{R}}),(\tau,d)),d_{\sup})$ is separable.

PROOF: let α be in $\sigma_{2\text{mon}}$. Consider the subset A of $[0,1]$ given by:

$A=\bigcup_{n\in\mathbb{N}}\{\beta\in[0,\frac{1}{2}-2^{-n}]\cup[\frac{1}{2}+2^{-n},1] \mid n\in\mathbb{N}\wedge\alpha(n)=0\}$

and let d be the metric $d_{\mathbb{R}}$. Then (A,d) is precompact (even fanlike). Let (τ,d) be the completion $\overline{(A,d)}$ of (A,d). Now suppose $(C(([0,1],d_{\mathbb{R}}),(\tau,d)),d_{\sup})$ is separable. Then we have a dense sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $(C(([0,1],d_{\mathbb{R}}),(\tau,d)),d_{\sup})$. We find:

$(\star)$ $\forall n\in\mathbb{N}\ \exists s\in\{0,1\}\ [\,(s=0\wedge d_{\sup}(\gamma_n,id_{[0,1]})>\frac{1}{3})\vee(s=1\wedge d_{\sup}(\gamma_n,id_{[0,1]})<\frac{1}{4})\,]$

By $\mathbf{AC}_{00}$ there is a β in σ_2 realizing $(\star)$. We then find: $\forall n\in\mathbb{N}\ [\alpha(n)=0]$ iff $\exists n\in\mathbb{N}\,[\beta(n)=1]$. Now suppose that for every metric fan (τ,d): $(C(([0,1],d_{\mathbb{R}}),(\tau,d)),d_{\sup})$ is separable. Since α above is arbitrary we obtain:

$(\star\star)$ $\forall\alpha\in\sigma_{2\text{mon}}\ \exists\beta\in\sigma_2\ [\,\alpha=\underline{0}\leftrightarrows\beta\#\underline{0}\,]$

By $\mathbf{AC}_{11}$ there is a spread-function γ from $\sigma_{2\text{mon}}$ to σ_2 realizing $(\star\star)$. Obviously $\gamma(\underline{0})\#\underline{0}$. But then there is an $n\in\mathbb{N}$ such that for all δ in $\sigma_{2\text{mon}}\cap\underline{\overline{0}}(n)$: $\gamma(\delta)\#\underline{0}$. This means that for all δ in $\sigma_{2\text{mon}}\cap\underline{\overline{0}}(n)$: $\delta=\underline{0}$ since γ realizes $(\star\star)$. Contradiction •

0.5.6 we can salvage the situation described above if (Y,d_Y) is a locally convex linear space.

THEOREM: let (τ,d) be a metric fan and let (L,d_L) be a locally convex linear space. Then the space $(C((\tau,d),(L,d_L)),d_{\sup})$ of all uniformly continuous spread-functions from (τ,d) to (L,d_L) (see 0.5.4) is a linear space.

PROOF: the only concern is that $(C((\tau,d),(L,d_L)),d_{\sup})$ be separable. This can be seen as follows. Let $(z_n)_{n\in\mathbb{N}}$ be dense in (L,d_L). Since (τ,d) is precompact we can find a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in τ such that:

$\forall n\in\mathbb{N}\ \exists m\in\mathbb{N}\ \forall\alpha\in\tau\ [\,\exists i\leq m\,[d(\alpha,\alpha_i)<2^{-n}]\,]$

For each $s,M\in\mathbb{N}$, and each function $\pi:\{0,\ldots,M\}\to\mathbb{N}$ we define a uniformly continuous function $f_{s,M,\pi}$ from (τ,d) to (L,d_L) by putting, for α in τ:

$f_{s,M,\pi}(\alpha)=\frac{\sum_{i\leq M}\sup(0,2^{-s}-d(x,\alpha_i))\cdot z_{\pi(i)}}{\sum_{j\leq M}\sup(0,2^{-s}-d(x,\alpha_j))}$

$\boxed{\text{claim}}$ $\{f_{s,M,\pi} \mid s, M \in \mathbb{N},\ \pi : \{0, \ldots, M\} \to \mathbb{N}\}$ is dense in $(C((\tau, d), (L, d_L)), d_{\sup})$.

$\boxed{\text{proof}}$ let f be in $C((\tau, d), (L, d_L))$, and let $n \in \mathbb{N}$ be arbitrary. It suffices to come up with $s, M \in \mathbb{N}$ and a $\pi : \{0, \ldots, M\} \to \mathbb{N}$ such that $d_{\sup}(f, f_{s,M,\pi}) < 2^{-n}$. Well, since (L, d_L) is locally convex we have:

$(\star)$ $\forall \alpha \in \tau\ \exists s, t \in \mathbb{N}\ [\, f(B(\alpha, 2^{-s})) \subset B(f(\alpha), 2^{-t-1}) \wedge conv(B(f(\alpha), 2^{-t})) \subset B(f(\alpha), 2^{-n-1})\,]$

By the fan theorem **FT** there are $S, T \in \mathbb{N}$ such that $f(B(\alpha, 2^{-S})) \subset B(f(\alpha), 2^{-T-1})$ and $conv(B(f(\alpha), 2^{-T})) \subset B(f(\alpha), 2^{-n-1})$ for all α in τ. Determine $M \in \mathbb{N}$ such that for all α in τ there is an $i \leq M$ with $d(\alpha, \alpha_i) < 2^{-S}$. Determine a function $\pi : \{0, \ldots, M\} \to \mathbb{N}$ such that $d_L(f(\alpha_i), z_{\pi(i)}) < 2^{-T-1}$ for all $i \leq M$. We hold: $d_{\sup}(f, f_{S,M,\pi}) < 2^{-n}$. For let α be arbitrary in τ. It suffices to show that $d_L(f(\alpha), f_{S,M,\pi}(\alpha)) \leq 2^{-n-1}$. By definition we have:

$$f_{S,M,\pi}(\alpha) = \frac{\sum_{i \leq M} \sup(0, 2^{-S} - d(x, \alpha_i)) \cdot z_{\pi(i)}}{\sum_{j \leq M} \sup(0, 2^{-S} - d(x, \alpha_j))}$$

Therefore $f_{S,M,\pi}(\alpha)$ is a convex combination of the $z_{\pi(i)}$'s and in fact it is a limit of convex combinations of $z_{\pi(i)}$'s for i's such that $d(\alpha, \alpha_i) < 2^{-S}$. But for i such that $d(\alpha, \alpha_i) < 2^{-S}$ we see that $d_L(f(\alpha), f(\alpha_i)) < 2^{-T-1}$ and $d_L(f(\alpha_i), z_{\pi(i)}) < 2^{-T-1}$, so $d_L(f(\alpha), z_{\pi(i)}) < 2^{-T}$. Therefore $f_{S,M,\pi}(\alpha)$ is a limit of convex combinations of elements of $B(f(\alpha), 2^{-T})$, and so $f_{S,M,\pi}(\alpha)$ is in $cB(f(\alpha), 2^{-n-1})$. Clearly then $d_L(f(\alpha), f_{S,M,\pi}(\alpha)) \leq 2^{-n-1}$ ∘•

COROLLARY: let $(L, \| \,\|)$ be a Banach space. Then $(C((\tau, d), (L, \| \,\|)), \| \,\|_{\sup})$ is a Banach space.

REMARK: using the fan theorem **FT** we can prove that if (L, d_L) is a locally convex linear space, then $(C((\tau, d), (L, d_L)), d_{\sup})$ is also locally convex.

0.5.7* a function space (a space of continuous functions from one metric space to another) is more easily studied when it is endowed with a metric such that two functions are metrically apart iff there is a point in which these functions assume different values. The metric $d_{\sup}$ discussed above is an obvious example, but it works only for metric fans. We discuss another such metric on the space of continuous spread-functions from a metric spread to another metric space.

DEFINITION: let (σ, d) be a metric spread, and let (Y, d_Y) be a metric space. We define a metric d_{dense} on the space $C((\sigma, d), (Y, d_Y))$ of all continuous spread-functions from (σ, d) to (Y, d_Y) by putting, for f and g in $C((\sigma, d), (Y, d_Y))$:

$d_{\text{dense}}(f,g) \underset{D}{=} \sum_{a\in\overline{\overline{\sigma}}} 2^{-a} \cdot \frac{d_Y(f(\alpha_a),g(\alpha_a))}{1+d_Y(f(\alpha_a),g(\alpha_a))}$

THEOREM: let (σ,d) be a metric spread and let (L,d_L) be a linear space. Then the space $(C((\sigma,d),(L,d_L)),d_{\text{dense}})$ of all continuous spread-functions from (σ,d) to (L,d_L) is a linear space.

PROOF: the only concern is that $(C((\sigma,d),(L,d_L)),d_{\text{dense}})$ be separable. This can be seen as follows. Let h be the unique enumeration of $\overline{\overline{\sigma}}$ (that is: $\overline{\overline{\sigma}}=\{h(n)\,|\,n\in\mathbb{N}\}$) such that $n<m$ iff $h(n)<h(m)$ for $n,m\in\mathbb{N}$ .We have:

($\star$) $\forall m\in\mathbb{N}\ \forall i,j\in\mathbb{N}\ \exists s\in\{0,1\}\ [\,(s=0 \wedge d(\alpha_{h(i)},\alpha_{h(j)}<2^{-m}) \vee (s=1 \wedge \alpha_{h(i)} \# \alpha_{h(j)})\,]$

By $\mathbf{AC}_{00}$ there is a function g from $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ realizing ($\star$). For $m,i,j\in\mathbb{N}$ we put, in order to get rid of symmetry problems, $e(m,i,j)=\min(g(m,i,j),g(m,j,i))$. For $m,i\in\mathbb{N}$ we put: $k(m,i)=\mu j\leq i\,[e(m,i,j)=0]$ (notice that $e(m,i,i)=0$ for all $m,i\in\mathbb{N}$). Then if $i=k(m,i)$ for certain $m,i\in\mathbb{N}$, we are sure that $\alpha_{h(i)} \# \alpha_{h(j)}$ for all $j<i$.

Now let $(z_n)_{n\in\mathbb{N}}$ be dense in (L,d_L) . For each $n,m\in\mathbb{N}$ and each function π from $\{0,\ldots,n\}$ to $\mathbb{N}$ we define a continuous function $f_{n,m,\pi}$ from (σ,d) to (L,d_L) by putting, for α in σ :

$f_{n,m,\pi}(\alpha) = \sum_{i\leq n, i=k(m,i)}\left(\frac{\prod_{j\leq n,\, e(m,i,j)=1} d(\alpha,\alpha_{h(j)})}{\prod_{j\leq n,\, e(m,i,j)=1} d(\alpha_{h(i)},\alpha_{h(j)})} \cdot z_{\pi(i)}\right)$

Then $\{f_{n,m,\pi} \mid n,m\in\mathbb{N},\ \pi:\ \{0,\ldots,n\}\to\mathbb{N}\}$ is a set of interpolation functions. We leave it to the reader to verify that this set is dense in $(C((\sigma,d),(L,d_L)),d_{\text{dense}})$ •

CHAPTER ONE

GENERAL TOPOLOGY

abstract

examining the idea of a 'topological space' we discover that every apartness induces a topology, called the *apartness topology*. Our analysis leads us to restrict our attention to *effective* separable spaces whose topology induces a so-called Σ^1_0-apartness. For this 'class' we define a large variety of topological notions such as 'continuous function', 'Hausdorff', 'connected', etc. Every function from an apartness space to another topological space is continuous. We define a topological space to be *compact* iff it is *fanlike* and Hausdorff. This closely parallels the classical definition. The topology $\mathcal{T}$ of a compact space $(X, \mathcal{T})$ is seen to be the apartness topology. This yields that every function from a compact space to another topological space is continuous. We define the subspace topology which is naturally carried by *separable* subsets of a topological space. We pay special attention to 'local' properties, in order to deal with the testcase 'locally compact'. The topology $\mathcal{T}$ of a locally compact space $(X, \mathcal{T})$ is also seen to be the apartness topology. Therefore every function from a locally compact space to another topological space is continuous.

1.0 DEFINITION OF 'TOPOLOGICAL SPACE'

1.0.0* we begin with a half formal, half informal discussion, which we hope will motivate certain restrictions we impose on our field of research, as well as clarify our definitions. We define a *topological space* as a pair $(X, \mathcal{T})$, where X is a subset of σ_ω, and $\mathcal{T}$ is a collection of subsets of X satisfying:

O_1. X and $\emptyset$ are in $\mathcal{T}$.

O_2. if A is a subset of X such that for all a in A there is a U in $\mathcal{T}$ such that $a \in U \subseteq A$, then A is in $\mathcal{T}$.

O_3. for all U, V in $\mathcal{T}$: $U \cap V$ is in $\mathcal{T}$.

Then $\mathcal{T}$ is called a *topology* on X. Let us agree to call $(X, \mathcal{T})$ *effective* iff:

O_4. $\forall x \in X\ \forall U \ni x\ \forall y \in X\ [\, y \in U \vee \exists V \ni x\, [y \notin V]\,]$

where we use 'U','V' for elements of $\mathcal{T}$. We will restrict our attention to effective topological spaces.

DEFINITION: let (X, d) be a metric space, let A be a subset of X. We say that A is *open* in (X, d) iff for all x in U there is an $n \in \mathbb{N}$ such that $B(x, 2^{-n}) \subseteq U$.

REMARK: notice that if A is open in (X, d), then A is closed under $\#_d$-equivalence, therefore A is a subset of (X, d).

Given a metric space (X, d) we can define a topology on X by letting $\mathcal{T}_d$ be the collection of open subsets of (X, d). Then $(X, \mathcal{T}_d)$ is an effective topological space, as is easily verified. We call $\mathcal{T}_d$ the *metric topology* on (X, d). We will simply write (X, d) for $(X, \mathcal{T}_d)$.

1.0.1* DEFINITION: let X be a subset of σ_ω, and let $\mathcal{T}_1$, $\mathcal{T}_2$ be two topologies on X. We say that $(X, \mathcal{T}_1)$ *refines* $(X, \mathcal{T}_2)$ iff every U in $\mathcal{T}_2$ is in $\mathcal{T}_1$. Then we also say that $\mathcal{T}_1$ is *finer* than $\mathcal{T}_2$. We say that $(X, \mathcal{T}_1)$ *coincides identically* with $(X, \mathcal{T}_2)$ iff $(X, \mathcal{T}_1)$ refines $(X, \mathcal{T}_2)$ and $(X, \mathcal{T}_2)$ refines $(X, \mathcal{T}_1)$.

1.0.2* DEFINITION: let $(X, \#)$ be an apartness space. Let A be a subset of X. Then A is *open* in $(X, \#)$ iff for all x in A we have: $\forall y \in X\ [\, y \in A \vee y \# x\,]$.

REMARK: notice that if A is open in $(X, \#)$, then A is closed under $\#$-equivalence, therefore A is a subset of $(X, \#)$ (see 0.1.1).

Given an apartness space $(X, \#)$ we can define a topology on X by letting $\mathcal{T}_\#$ be the collection of all subsets of X which are open in $(X, \#)$. We call $\mathcal{T}_\#$ the *apartness topology* on $(X, \#)$. We simply write $(X, \#)$ for $(X, \mathcal{T}_\#)$.

It is easy to verify that $(X, \#)$ satisfies O_1, O_2, and O_3. Notice that $(X, \#)$ is effective, that is, satisfies O_4. For let y be in X, then the set $V = \{y\}^\# = \{z \in X \mid z \# y\}$ is open in $(X, \#)$, since $\#$ is an apartness. Now let U be open in $(X, \#)$, and let x be in U. Then we have: $y \in U \vee y \# x$ which implies: $y \in U \vee x \in V \not\ni y$.

1.0.3* on the other hand, an effective topological space naturally carries an apartness, which we describe in the next definition.

DEFINITION: let $(X, \mathcal{T})$ be an effective topological space. Let x, y be in X, then $x \#_\mathcal{T} y$ iff $\exists U \in \mathcal{T}\ [x \in U \not\ni y \vee y \in U \not\ni x]$.

We verify that $\#_\mathcal{T}$ is an apartness: suppose $x \#_\mathcal{T} y$ and let z be in X. Without loss of generality, let U be in $\mathcal{T}$ such that $x \in U \not\ni y$. Since $\mathcal{T}$ is effective we can decide: $z \in U \not\ni y$ implying $z \#_\mathcal{T} y$, or $\exists V \ni x\, [z \notin V]]$ implying $z \#_\mathcal{T} x$.

We have: $(X, \#_\mathcal{T})$ refines $(X, \mathcal{T})$, since if U is in $\mathcal{T}$, then U is open in $(X, \#_\mathcal{T})$. For if x is in U and y is in X we obtain: $y \in U \vee y \#_\mathcal{T} x$ by O_4. Notice that for an apartness space $(X, \#)$ the apartnesses $\#$ and $\#_\mathcal{T}$ coincide.

On the third hand, by definition any apartness $\#$ on X is refined by the natural apartness $\#_\omega$ on σ_ω, restricted to X (meaning: $x \# y$ implies $x \#_\omega y$ for all x, y in X). Therefore the finest possible effective topology on X is the $\#_\omega$-topology $(X, \#_\omega)$. If we specify $X = \sigma$, a spread, then **CP** implies that d_ω metrizes $(\sigma, \#_\omega)$, which is the following lemma.

1.0.4 LEMMA: let σ be a spread. Then (σ, d_ω) coincides identically with $(\sigma, \#_\omega)$.

PROOF: trivially $(\sigma, \#_\omega)$ refines (σ, d_ω). Now let U be open in $(\sigma, \#_\omega)$, and let α in U. Then we have:

$$(\star) \quad \forall \beta \in \sigma\ \exists s \in \{0,1\}\ [(s = 0 \wedge \beta \in U) \vee (s = 1 \wedge \beta \# \alpha)]$$

By **CP** applied to α, there is an $n\in\mathbb{N}$ such that for all β in $\sigma\cap\overline{\alpha}(n)$: β is in U. α being arbitrary, this implies that U is open in (σ, d_ω) •

Therefore the finest possible effective topology on a spread σ is the metric topology induced by d_ω.

1.0.5* $(\sigma, \#_\omega)$ has a special property: $\#_\omega$ is induced by the decidable subset $\approx$ of $\overline{\overline{\sigma}}_\omega \times \overline{\overline{\sigma}}_\omega$ given by $\approx = \{(a,b)\in\overline{\overline{\sigma}}_\omega \times \overline{\overline{\sigma}}_\omega \,|\, a \sqsubseteq b \vee b \sqsubseteq a\}$. For if we denote the complement of $\approx$ by $\not\approx$, then for α, β in σ_ω: $\alpha \#_\omega \beta$ iff $\exists n\in\mathbb{N}\,[\,\overline{\alpha}(n) \not\approx \overline{\beta}(n)\,]$. The crucial aspect however is that '$\alpha \#_\omega \beta$' is determined in the course of time, popularly speaking. Therefore we define:

DEFINITION: let us call an apartness $\#$ on a subset X of σ_ω a Σ^1_0-*apartness* iff for all α, β in X there is a γ in $\sigma_{2\text{mon}}$ such that $\alpha \# \beta$ iff $\exists n\in\mathbb{N}\,[\,\gamma(n)=1\,]$. By extension we call $(X, \#)$ a Σ^1_0-apartness space.

By proposition 2.0.2 (using $\mathbf{AC}_{11}$) a Σ^1_0-apartness on a spread σ is in fact determined by a decidable subset $\approx$ of $\overline{\overline{\sigma}} \times \overline{\overline{\sigma}}$, and such that for α, β in σ: $\alpha \# \beta$ iff $\exists n\in\mathbb{N}\,[\,\overline{\alpha}(n) \not\approx \overline{\beta}(n)\,]$. We believe that Σ^1_0-apartnesses are the most natural and most manageable apartnesses. In accordance herewith, for a very large class of topological spaces $(X, \mathcal{T})$ the following holds:

O_5. $(X, \#_{\mathcal{T}})$ is a Σ^1_0-apartness space.

For a Σ^1_0-apartness spread $(\sigma, \#)$, the apartness topology is determined by the decidable subset $\approx$ of $\mathbb{N}$. But (σ, d_ω) has an even more special feature: an enumerable basis.

1.0.6* DEFINITION: let $(X, \mathcal{T})$ be a topological space and let $(U_n)_{n\in\mathbb{N}}$ be a sequence of inhabited elements of $\mathcal{T}$ satisfying: for all $x\in U\in\mathcal{T}$ there is $n\in\mathbb{N}$ such that $x\in U_n\subseteq U$. Then $(U_n)_{n\in\mathbb{N}}$ is called an *enumerable basis* of $(X, \mathcal{T})$.

REMARK: on the other hand let $(U_n)_{n\in\mathbb{N}}$ be a sequence of inhabited subsets of X such that $\bigcup_n U_n = X$ and for all $n, m\in\mathbb{N}$ and all x in $U_n \cap U_m$ there is an $s\in\mathbb{N}$ with $x\in U_s \subseteq U_n\cap U_m$. We can then define a subset A of X to be open iff for all a in A there is an $n\in\mathbb{N}$ with $a\in U_n\subseteq A$. The collection $\mathcal{T}$ of all such open sets is a topology on X, and $(U_n)_{n\in\mathbb{N}}$ is an enumerable basis of $(X, \mathcal{T})$. We then say that $(X, \mathcal{T})$ is *generated* by $(U_n)_{n\in\mathbb{N}}$.

So let us call $(X, \mathcal{T})$ *first-separable* iff

O_6. $(X, \mathcal{T})$ has an enumerable basis $(U_n)_{n\in\mathbb{N}}$.

and *second-separable* iff

O_7. there is a sequence $(a_n)_{n\in\mathbb{N}}$ of elements of X, which is dense in $(X,\mathcal{T})$, meaning: for all inhabited U in $\mathcal{T}$ there is $n\in\mathbb{N}$ such that $a_n \in U$.

It follows easily from $\mathbf{AC}_{01}$ that every first-separable space is second-separable. See example 1.4.4 for a topological space which is O_1 through O_5 and second-separable (O_7), but NOT first-separable (O_6). Notice that any apartness spread $(\sigma,\#)$ is second-separable, and that any second-separable metric space is first-separable. We do not however see a way to prove in general that a Σ^1_0-apartness spread is first-separable. Neither can we come up with a counterexample.

1.0.7* we hope that the previous discussion gives a fair motivation of the following convention.

CONVENTION: from now on when we write: 'let $(X,\mathcal{T})$ be a topological space' we tacitly assume that $(X,\mathcal{T})$ is an effective second-separable topological space such that in addition $(X, \#_{\mathcal{T}})$ is a Σ^1_0-apartness space.

In other words, we restrict ourselves to spaces satisfying O_1, O_2, O_3, O_4, O_5 and O_7. We therefore define topological notions such as 'continuous', 'connected', etc. only for the above mentioned spaces. Among these spaces the following stand out:

(i) first-separable spaces $(X,\mathcal{T})$, where X is enumerable.

(ii) Σ^1_0-apartness spreads.

(iii) second-separable metric spaces.

REMARK: ad (i): these spaces are also of interest from a combinatorial point of view. Refining their topologies by $(X, \#_{\mathcal{T}})$ one loses the essential features. Let us agree to call an $(X,\mathcal{T})$ satisfying (i) above a *first-enumerable* space. In practice our first-enumerable spaces will have an enumerable basis of decidable subsets of $\mathbb{N}$.

1.0.8* we illustrate convention 1.0.7 with a few examples:

EXAMPLE: the *discrete* topology on σ_ω 'simply' contains 'all subsets of σ_ω', and is clearly NOT effective (use $\mathbf{CP}$).

EXAMPLE: the *needle topology* on $\mathbb{R}$ is generated by the countable basis $\{[p,q)\,|\,p,q\in\mathbb{Q},\ p<q\}$. The needle topology refines the metric topology $(\mathbb{R},d_{\mathbb{R}})$, but

it is NOT an effective topology, which we see as follows. Let $p \in \mathbb{Q}$, then p is in $U = [p, p+1)$. Suppose the needle topology is effective, then for all $x \in \mathbb{R}$ we can decide: $x \in [p, p+1)$ or $x \#_{\mathbb{R}} p$. This means that for all $x \in \mathbb{R}$ we can decide: $x < p$ or $x \geq p$. Contradiction, see 0.2.2.

EXAMPLE: the *halfline topology* $\mathcal{T}_h$ on $\mathbb{R}$ is generated by the countable basis $\{\{\alpha \in \mathbb{R} \mid \alpha > p\} \mid p \in \mathbb{Q}\}$. In contrast to the previous examples, this is an effective topology. Since the corresponding $\#_{\mathcal{T}}$ is simply $\#_{\mathbb{R}}$ $(\mathbb{R}, \mathcal{T}_h)$ satisfies O_1 through O_7. $(\mathbb{R}, \mathcal{T}_h)$ is of course refined by the metric topology $(\mathbb{R}, d_{\mathbb{R}})$.

EXAMPLE: an *enumerable graph* is a pair (X, E) where X is an enumerable subset of σ_ω, and E is a decidable subset of $X \times X$. Let (X, E) be an enumerable graph. Let A be a subset of $X \cup E$. A is *open in* (X, E) iff for all $x \in A \cap X$: $\{e \in E \mid \exists y \in X\, [(x, y) = e]\} \subseteq A$. The collection $\mathcal{T}_{(X,E)}$ of all open sets in (X, E) is called the *graph-topology* on (X, E). We simply write (X, E) for $(X \cup E, \mathcal{T}_{(X,E)})$. In this context we will say that (X, E) is a *topological graph*. A topological graph (X, E) is first-enumerable, but there are many first-enumerable spaces which do not coincide with a topological graph, see for instance examples 2.0.3 and 2.0.4.

1.0.9* a more important example is the following one, which shows that our class of topological spaces is closed under the operation of taking infinite products.

EXAMPLE: let $(X_n, \mathcal{T}_n)_{n \in \mathbb{N}}$ be a sequence of topological spaces satisfying O_1 through O_5 and O_7. Since for each $n \in \mathbb{N}$ X_n is a subset of σ_ω, we can form the product $\Pi_{n \in \mathbb{N}} X_n = \{\alpha \in \sigma_\omega \mid \forall n \in \mathbb{N}\, [\alpha_{[n]} \in X_n]\}$ as a subset of σ_ω. Then the *product topology* $\mathcal{T}_{\text{prod}}$ on $\Pi_{n \in \mathbb{N}} X_n$ is defined by declaring a subset A of $\Pi_{n \in \mathbb{N}} X_n$ to belong to $\mathcal{T}_{\text{prod}}$ iff for all $n \in \mathbb{N}$: $\{\alpha_{[n]} \mid \alpha \in A\}$ is open in $(X_n, \mathcal{T}_n)$ and in addition there is an $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}, m > N$: $\{\alpha_{[m]} \mid \alpha \in A\} = X_m$.

This is completely analogous to the classical definition. We leave it to the reader to verify that $(\Pi_{n \in \mathbb{N}} X_n, \mathcal{T}_{\text{prod}})$ satisfies O_1 through O_5 and O_7. $(\Pi_{n \in \mathbb{N}} X_n, \mathcal{T}_{\text{prod}})$ satisfies O_6 iff for each $n \in \mathbb{N}$: $(X_n, \mathcal{T}_n)$ satisfies O_6. The same holds mutatis mutandis for 'metrizable'. We also write $\Pi_{n \in \mathbb{N}}(X_n, \mathcal{T}_n)_{n \in \mathbb{N}}$ for $(\Pi_{n \in \mathbb{N}} X_n, \mathcal{T}_{\text{prod}})$. Notice that $(\sigma_\omega, \#_\omega)$ coincides with $\Pi_{n \in \mathbb{N}}(\mathbb{N}, \#_\omega)$, and $(\mathbb{R}^{\mathbb{N}}, \#_{\mathbb{R}^{\mathbb{N}}})$ coincides with $\Pi_{n \in \mathbb{N}}(\mathbb{R}, \#_{\mathbb{R}})$.

1.0.10* CONVENTION: for apartness and metric spaces we write $(X, \#), (X, d)$ rather than $(X, \mathcal{T}_\#)$, $(X, \mathcal{T}_d)$. Elements of $\mathcal{T}$ will be called *open* in $(X, \mathcal{T})$. For $\#_{\mathcal{T}}$ we simply write $\#$ whenever confusion is unlikely to occur.

1.1 (NOT SO) COMMON DEFINITIONS

1.1.0* DEFINITION: let $(X,\mathcal{T})$, $(Y,\mathcal{T}')$ be topological spaces, and let $\#$, $\#_Y$ be the apartnesses induced by $\mathcal{T}$, $\mathcal{T}'$ respectively, via definition 1.0.3. Let f be a (weak) function from $(X,\#)$ to $(Y,\#_Y)$ (see 0.1.3). Then f is a *(weak) function* from $(X,\mathcal{T})$ to $(Y,\mathcal{T}')$, notation $f:(X,\mathcal{T})\to(Y,\mathcal{T}')$. Similarly, if A is a subset of $(X,\#)$, then A is a *subset* of $(X,\mathcal{T})$. Now let f be a weak function from $(X,\mathcal{T})$ to $(Y,\mathcal{T}')$. Then f is *$(\mathcal{T},\mathcal{T}')$-continuous* iff for all V in $\mathcal{T}'$: $f^{-1}(V)$ is in $\mathcal{T}$. When the context is clear we simply write 'f is continuous'. We say that $(Y,\mathcal{T}')$ is a *continuous image* of $(X,\mathcal{T})$ iff there is a continuous surjection from $(X,\mathcal{T})$ to $(Y,\mathcal{T}')$.

REMARK: for a first-separable $(Y,\mathcal{T}')$ with basis $(V_n)_{n\in\mathbb{N}}$ we have: f is continuous iff for all $n\in\mathbb{N}$: $f^{-1}(V_n)$ is in $\mathcal{T}$. For apartness spaces we have the following important theorem.

THEOREM: every function from an apartness space to another topological space is continuous.

PROOF: let $(X,\#)$ be an apartness space, and let f be a function from $(X,\#)$ to a topological space $(Y,\mathcal{T}')$. Let V be in $\mathcal{T}'$, and let x be in $f^{-1}(V)$. Let z be in X. Since $\mathcal{T}'$ is effective we find: $f(z)\in V\vee f(z)\#_Y f(x)$. This implies: $z\in f^{-1}(V)\vee z\# x$ since f is a function. Therefore $f^{-1}(V)$ is open in $(X,\#)$ •

REMARK: we will reduce a famous intuitionistic result, namely the continuity of everywhere defined real functions (see 0.4.1), to this theorem and **CP**, see 3.3.10. Conversely we have:

LEMMA: let f be a continuous weak function from $(X,\mathcal{T})$ to $(Y,\mathcal{T}')$. Then f is a function from $(X,\mathcal{T})$ to $(Y,\mathcal{T}')$.

PROOF: let x, y be in X such that $f(x)\#_Y f(y)$. Then without loss of generality there is a V in $\mathcal{T}'$ such that $f(x)\in V\not\ni f(y)$. Then $x\in f^{-1}(V)\not\ni y$, and $f^{-1}(V)$ is in $\mathcal{T}$ by the continuity of f. Therefore $x\# y$ •

1.1.1* DEFINITION: let $(X,\mathcal{T})$, $(Y,\mathcal{T}')$ be topological spaces, and let h be a continuous function from $(X,\mathcal{T})$ to $(Y,\mathcal{T}')$. Then h is called a *homeomorphism* from $(X,\mathcal{T})$ to $(Y,\mathcal{T}')$ iff $h^{-1}=\{(y,x)\in Y\times X \mid h(x)\equiv_Y y\}$ is a continuous function from $(Y,\mathcal{T}')$ to $(X,\mathcal{T})$. In this case h^{-1} is called the *inverse homeomorphism* of h. We say that $(X,\mathcal{T})$ *coincides* with $(Y, \mathcal{T}')$ iff there is a homeomorphism h from $(X, \mathcal{T})$ to $(Y, \mathcal{T}')$.

REMARK: notice that given two topologies $\mathcal{T}$, $\mathcal{T}_1$ on X, a subset of σ_ω, we have that $(X, \mathcal{T})$ coincides identically with $(X, \mathcal{T}_1)$ (definition 1.0.1) iff the identity is a homeomorphism from $(X, \mathcal{T})$ to $(X, \mathcal{T}_1)$. Also, if h is a homeomorphism from $(X, \mathcal{T})$ to $(Y, \mathcal{T}')$, then by definition we have: $h^{-1} \circ h = id_X$ and $h \circ h^{-1} = id_Y$.

LEMMA:

(i) let h be an injective and surjective function from $(X, \#)$ to $(Y, \#_Y)$, two apartness spaces. Then h is a homeomorphism from $(X, \#)$ to $(Y, \#_Y)$.

(ii) let $(X, \mathcal{T})$ be a topological space coinciding with an apartness space $(Y, \#_Y)$. Then $(X, \mathcal{T})$ coincides identically with $(X, \#)$.

PROOF: ad (i): that h^{-1} is a function follows from the injectivity and surjectivity of h. Both h and h^{-1} are continuous by theorem 1.1.0. For (ii) let h be a homeomorphism from $(X, \mathcal{T})$ to $(Y, \#_Y)$, with inverse h^{-1}. Then h^{-1} is a continuous function from $(Y, \#_Y)$ to $(X, \#)$ by theorem 1.1.0. Therefore $h^{-1} \circ h = id_X$ is a homeomorphism from $(X, \mathcal{T})$ to $(X, \#)$ •

From a topological point of view homeomorphic spaces are identical. Thus it is natural to primarily study properties, relations, etc. which are *topological*, that is: invariant under homeomorphisms. An important example of a non-topological concept is the completeness of a metric space (X, d). The following definition is the conventional remedy.

1.1.2* DEFINITION: let $(X, \mathcal{T})$ be a topological space, then $(X, \mathcal{T})$ is *topologically complete* iff $(X, \mathcal{T})$ coincides with a complete metric space (Y, d_Y).

Similarly there is a cheap way to 'topologize' any concept C: $(X, \mathcal{T})$ is 'topologically C' iff $(X, \mathcal{T})$ coincides with a $(Y, \mathcal{T}')$ which is C. Frequently we are interested in an alternative characterization of 'topologically C'.

1.1.3* DEFINITION: a topological space $(X, \mathcal{T})$ is called *metrizable* iff $(X, \mathcal{T})$ coincides with a metric space (Y, d_Y). A topological space $(X, \mathcal{T})$ is called *weakly metrizable* iff there is a metric space (Y, d_Y) such that $(X, \#)$ coincides with $(Y, \#_Y)$.

REMARK: $(X, \mathcal{T})$ is metrizable iff there is a metric d on X such that the identity is a homeomorphism from $(X, \mathcal{T})$ to (X, d). $(X, \mathcal{T})$ is weakly metrizable iff there is a metric d on

X such that for all x, y in X: $x \# y$ iff $d(x,y)>0$ (meaning the identity is a homeomorphism from $(X, \#)$ to $(X, \#_d)$). This is easily seen by defining $d(x,y)=d_Y(h(x),h(y))$ ($x,y \in X$) for a given homeomorphism h from $(X,\mathcal{T})$ to (Y,d_Y).

In the same vein: if σ is a spread, and f is a surjection from (σ,d_ω) to a metric space (X,d), then (X,d) coincides with the metric spread (σ,d_σ), where $d_\sigma(\alpha,\beta)=d(f(\alpha),f(\beta))$ for α, β in σ. But (σ,d_ω) is the continuous image of (σ_ω,d_ω) under the canonical retraction $\pi_{\omega,\sigma}$ defined in 0.0.3, therefore (σ,d_σ) is the continuous image of (σ_ω,d_ω). Therefore (σ,d_σ) coincides with (σ_ω,d) for a metric d on σ_ω obtained as above. This shows that in fact a metric spread (σ,d_σ) is 'nothing but' a metric d on σ_ω.

1.1.4* DEFINITION: let $(X,\mathcal{T})$ be a topological space, and let $\mathcal{U}$ be a collection of elements of $\mathcal{T}$. Then $\mathcal{U}$ is called a *open cover* of $(X,\mathcal{T})$ iff for all x in X there is a U in $\mathcal{U}$ such that x is in U. Now let $\mathcal{U}$, $\mathcal{V}$ be open covers of $(X,\mathcal{T})$. $\mathcal{V}$ is called a *refinement* of $\mathcal{U}$ iff for each V in $\mathcal{V}$ there is a U in $\mathcal{U}$ such that $V \subseteq U$. $\mathcal{V}$ is called a *subcover* of $\mathcal{U}$ iff each V in $\mathcal{V}$ is in $\mathcal{U}$.

1.1.5* DEFINITION: a topological space $(X,\mathcal{T})$ is called *spreadlike* iff there is a continuous surjection from (σ_ω,d_ω) to $(X,\mathcal{T})$. $(X,\mathcal{T})$ is *fanlike* iff there is a continuous surjection from (σ_2,d_ω) to $(X,\mathcal{T})$.

REMARK:

(i) any spread (σ,d_ω) is the continuous image of (σ_ω,d_ω) under the canonical retraction $\pi_{\omega,\sigma}$ defined in 0.0.3. So if there is a continuous surjection from (σ,d_ω) to $(X,\mathcal{T})$, then $(X,\mathcal{T})$ is spreadlike, and fanlike if σ is a fan. This shows that $(X,\mathcal{T})$ is spreadlike iff $(X,\mathcal{T})$ coincides with a topological spread $(\sigma,\mathcal{T}_\sigma)$. For we can pull back the topology on X to a topology on σ using the given surjection. Simply let A be in $\mathcal{T}_\sigma$ iff $f(A)$ is in $\mathcal{T}$ and $A=f^{-1}(f(A))$. It is straightforward to check that O_1 through O_7 are preserved under this pulling back. Similarly $(X,\mathcal{T})$ is fanlike iff $(X,\mathcal{T})$ coincides with a topological fan $(\sigma,\mathcal{T}_\sigma)$.

(ii) in view of (i) we prefer '$(X,\mathcal{T})$ is spreadlike' to the classically equivalent terminology 'X is analytical'.

(iii) it might be wise to restrict our attention to spreadlike topological spaces. But certain interesting metric spaces, notably spaces of continuous functions, are not spreadlike. Therefore we will not adopt such limitation, although it would render certain definitions and theorems less tiresome.

1.1.6 DEFINITION: let us call $(X,\mathcal{T})$ *Lindelöf* iff each open cover of $(X,\mathcal{T})$ has an enumerable refinement.

PROPOSITION: let $(\sigma,\mathcal{T})$ be a topological spread, and suppose $(U_n)_{n\in\mathbb{N}}$ is an enumerable basis of $(\sigma,\mathcal{T})$. Let $\mathcal{U}$ be an open cover of $(\sigma,\mathcal{T})$. Then there is a γ in σ_ω such that $\{U_{\gamma(n)} \mid n\in\mathbb{N}\}$ is a refinement of $\mathcal{U}$.

PROOF: let $\mathcal{U}$ be an open cover of $(\sigma,\mathcal{T})$. Since $(U_n)_{n\in\mathbb{N}}$ is an enumerable basis of $(\sigma,\mathcal{T})$, we find:

$$(\star)\quad \forall\alpha\in\sigma\ \exists n\in\mathbb{N}\ [\,\alpha\in U_n \wedge \exists U\in\mathcal{U}\,[U_n\subseteq U]\,]$$

By $\mathbf{AC}_{10}$ there is a spread-function γ from σ to $\mathbb{N}$ realizing ($\star$). Put $D=\{d\in\overline{\overline{\sigma}}\,|\,\gamma(d)\neq 0\}$. Then D is a decidable inhabited subset of $\mathbb{N}$, so D is enumerable. Let γ be in σ_ω such that $D=\{\gamma(n)\,|\,n\in\mathbb{N}\}$. Then $\{U_{\gamma(n)} \mid n\in\mathbb{N}\}$ is a refinement of $\mathcal{U}$ •

COROLLARY: every first-separable spreadlike topological space is Lindelöf.

REMARK: in fact we do not have any example of a Lindelöf $(X,\mathcal{T})$ which is not spreadlike. For metric spreads this corollary, with similar proof, occurs already in [Troelstra66].

1.1.7* DEFINITION: let $(X,\mathcal{T})$ be a topological space. We define the following separation properties for $(X,\mathcal{T})$:

T_0. $\forall x,y\in X\ [\,x\,\#_{\mathcal{T}}\,y \rightarrow \exists U\in\mathcal{T}\,[x\in U\not\ni y \vee y\in U\not\ni x]\,]$.

T_1. $\forall x,y\in X\ [\,x\,\#_{\mathcal{T}}\,y \rightarrow \exists U\in\mathcal{T}\,[x\in U\not\ni y]\,]$.

T_2. $\forall x,y\in X\ [\,x\,\#_{\mathcal{T}}\,y \rightarrow \exists U,V\in\mathcal{T}\,[x\in U \wedge y\in V \wedge U\cap V=\emptyset]$.

T_3. $\forall x\in X\,\forall U\ni x\ \exists V\ni x,\ W\in\mathcal{T}\ [\,U\cup W=X \wedge V\cap W=\emptyset\,]$.

T_4. $\forall U,V\in\mathcal{T}\ [\,U\cup V=X \rightarrow \exists W,Z\in\mathcal{T}\,[U\cup W=X=V\cup Z \wedge W\cap Z=\emptyset]\,]$.

We will simply write: '$(X,\mathcal{T})$ is T_2' etcetera. A space which is T_2 is usually called *Hausdorff*, T_3 goes by the name of *regular* and the combination of T_1 and T_4 listens to the endearment *normal*. For first-enumerable spaces we obtain one alternative separation property $\mathrm{T}_{0,e}$ by replacing $\#_{\mathcal{T}}$ with $\#_\omega$ in T_0.

REMARK: notice that any $(X,\mathcal{T})$ is T_0, by definition of $\#_{\mathcal{T}}$. Therefore only $\mathrm{T}_{0,e}$ is an interesting property, which expresses that $\#_\omega$ equals $\#_{\mathcal{T}}$. Classically each metric space is normal. Intuitionistically also a metric spread (σ, d) is normal, see theorem 3.1.5. The proof is not so easy as the classical proof of the classical theorem. On the other hand it is not difficult to see that every metric space (X, d) is regular.

LEMMA: a topological space $(X, \mathcal{T})$ is T_1 iff for all x in X: $\{x\}^{\#} = \{y \in X \mid y \# x\}$ is open in $(X, \mathcal{T})$.

PROOF: let $(X, \mathcal{T})$ be T_1, and let x be in X, y in $\{x\}^{\#}$. Then there is a U in $\mathcal{T}$ such that $y \in U \not\ni x$. Then for all z in U: $z \# x$ so z is in $\{x\}^{\#}$. Therefore $U \subseteq \{x\}^{\#}$ and so, by O_2, $\{x\}^{\#}$ is open in $(X, \mathcal{T})$. The other implication is trivial •

COROLLARY: every apartness space is T_1. (See 1.0.2).

EXAMPLE: we give rather trivial examples of spaces $(X, \mathcal{T})$ which are (i) not $T_{0,e}$ (ii) $T_{0,e}$ and T_4, but not T_3. For (i) let $X = \{0,1\}$ and $\mathcal{T} = \{\emptyset, \{0,1\}\}$. For (ii) let $X = \{0,1\}$ and $\mathcal{T} = \{\emptyset, \{1\}, \{0,1\}\}$.

More interesting examples will appear in our study of Σ^1_0-apartness spreads. Notwithstanding example (ii) above we have:

PROPOSITION:

(i) every normal space is regular.

(ii) every regular space is Hausdorff.

(iii) every Hausdorff space is T_1.

PROOF: let $(X, \mathcal{T})$ be a topological space. Ad (i): suppose $(X, \mathcal{T})$ is normal. Let $x \in U \in \mathcal{T}$. We must come up with an open $V \ni x$ and an open W such that $U \cup W = X$ and $V \cap W = \emptyset$. By the above lemma $\{x\}^{\#}$ is in $\mathcal{T}$. Also $U \cup \{x\}^{\#} = X$ since $(X, \mathcal{T})$ is effective. Using the normality of $(X, \mathcal{T})$, determine W, Z in $\mathcal{T}$ such that $U \cup W = X = \{x\}^{\#} \cup Z$ and $W \cap Z = \emptyset$. Clearly x is in Z, so we can take V equal to Z. Ad (ii): suppose $(X, \mathcal{T})$ is regular. Let x, y be in X such that $x \# y$. We must come up with an open $V \ni x$ and an open $W \ni y$ such that $V \cap W = \emptyset$. Without loss of generality let U in $\mathcal{T}$ such that $x \in U \not\ni y$. Using the regularity of $(X, \mathcal{T})$, determine $V \ni x$, $W \in \mathcal{T}$ such that $U \cup W = X$ and $V \cap W = \emptyset$. Clearly y is in W. (iii) is trivial •

1.1.8* DEFINITION: a topological space $(X, \mathcal{T})$ is called *connected* iff for all inhabited U, V in $\mathcal{T}$: $X = U \cup V$ implies $\exists x \in X\, [x \in U \cap V]$. $(X, \mathcal{T})$ is *pathwise connected* iff for all x, y

in X there is a continuous $f:([0,1],d_{\mathbb{R}})\to(X,\mathcal{T})$ such that $f(0)\equiv x$ and $f(1)\equiv y$. We say that $(X,\mathcal{T})$ is *arcwise connected* iff for all x,y in X, such an f can be found with the property that $x\#y$ iff f is injective.

1.2 COMPACT SPACES

1.2.0* one of the most important concepts in topology is that of a compact space. Classically a topological space $(X,\mathcal{T})$ is compact iff every open cover has a finite refinement; some authors add the condition that $(X,\mathcal{T})$ be Hausdorff. Some of the beauty of this classical concept lies in the following classical results. Firstly the continuous image of a compact space is again compact (for this the continuous image must be Hausdorff, if 'Hausdorff' is added to the definition of 'compact'). Secondly, if f is a continuous function from a compact space $(X,\mathcal{T})$ to $(\mathbb{R},d_{\mathbb{R}})$, then f assumes its maximum and its minimum. A fortiori $\sup(\{f(x)\,|\,x\in X\})$ and $\inf(\{f(x)\,|\,x\in X\})$ exist. Thirdly, if g is a continuous function from a compact metric space to another metric space, then g is uniformly continuous. Fourthly, if (X,d) is a compact metric space, then (X,d) is complete.

We cannot hope to recapture all of these classical results with an intuitionistic definition of 'compact'. We will make this clear in the examples of this section. But if we content ourselves with the existence of compact metric spaces which are not complete, then an attractive intuitionistic theory is possible. Also, a famous example of Brouwer shows that we cannot assert the existence of the maximum and minimum of an arbitrary continuous function from $([0,1],d_{\mathbb{R}})$ to $([0,1],d_{\mathbb{R}})$. It is possible though to compute the supremum and infimum of such a function.

In order to find an elegant intuitionistic definition of compact, we concentrate on the above issues. Our first try is a simple copy of the classical definition.

DEFINITION: a topological space $(X,\mathcal{T})$ is called *weakly compact* iff for each open cover $\mathcal{U}$ of $(X,\mathcal{T})$ there is a finite sequence $U_0,\ldots,U_K$ of elements of $\mathcal{U}$ such that $\{U_i\,|\,i\leq K\}$ is a cover of $(X,\mathcal{T})$.

LEMMA: let $(X,\mathcal{T})$ be a topological space, and let $\mathcal{U}$ be an open cover of $(X,\mathcal{T})$. Then there is a subcover $\mathcal{V}$ of $\mathcal{U}$ such that every V in $\mathcal{V}$ is inhabited.

PROOF: take $\mathcal{V}=\{V\in\mathcal{U}\mid V \text{ is inhabited}\}$. Trivially every V in $\mathcal{V}$ is an inhabited element of $\mathcal{U}$. To see that $\mathcal{V}$ is an open cover of $(X,\mathcal{T})$, let x be in X. Determine U in $\mathcal{U}$ such that x is in U. Then U is inhabited, therefore U is in $\mathcal{V}$ •

COROLLARY: let $(X,\mathcal{T})$ be a weakly compact space, and let $\mathcal{U}$ be an open cover of $(X,\mathcal{T})$. Then there is a finite sequence $U_0,\ldots,U_K$ of inhabited elements of $\mathcal{U}$ such that $\{U_i \,|\, i \leq K\}$ is a cover of $(X,\mathcal{T})$.

PROPOSITION:

(i) the continuous image of a weakly compact space is again weakly compact.

(ii) every continuous function from a weakly compact metric space (X,d) to another metric space (Y,d_Y) is uniformly continuous.

(iii) if $(X,d_{\mathbb{R}})$ is a weakly compact subspace of $(\mathbb{R},d_{\mathbb{R}})$, then $\sup(X)$ and $\inf(X)$ exist.

PROOF: (i) is trivial. For (ii) let $n \in \mathbb{N}$ be arbitrary. It suffices to prove that there is an $m \in \mathbb{N}$ such that for all x, y in X: if $d(x,y) < 2^{-m}$, then $d_Y(f(x),f(y)) < 2^{-n}$. Consider:

$$\mathcal{U} = \{B(x,2^{-s}) \mid x \in X, s \in \mathbb{N} \mid \forall y \in B(x,2^{-s})\,[d_Y(f(x),f(y)) < 2^{-n}]\}$$

Since f is continuous, $\mathcal{U}$ is an open cover of (X,d). Since (X,d) is weakly compact, we can find a finite sequence $U_0,\ldots,U_K$ of elements of $\mathcal{U}$ such that $\{U_i \,|\, i \leq K\}$ is a cover of (X,d). Using this finite sequence it is easy to find an $m \in \mathbb{N}$ with the desired property.

Finally (iii). Let $n \in \mathbb{N}$, and let $\mathcal{U}$ be the cover of $(X,d_{\mathbb{R}})$ given by: $\mathcal{U} = \{B(\frac{m}{2^n},2^{-n}) \cap X \mid m \in \mathbb{Z}\}$. By the previous corollary we can find a finite sequence $U_0,\ldots,U_K$ of inhabited elements of $\mathcal{U}$ such that $\{U_i \,|\, i \leq K\}$ is a cover of $(X,d_{\mathbb{R}})$. We now determine an $m \in \mathbb{Z}$ and a $j \leq K$ such that $U_j = B(\frac{m}{2^n},2^{-n}) \cap X$ and for all x in X: $x < \frac{m+1}{2^n}$. Clearly then $\frac{m}{2^n}$ is a 2^{-n}-accurate approximation of $\sup(X)$. Since $n \in \mathbb{N}$ is arbitrary, $\sup(X)$ exists. A very similar argument gives that $\inf(X)$ exists •

COROLLARY: let f be a continuous function from a weakly compact topological space $(X,\mathcal{T})$ to $(\mathbb{R},d_{\mathbb{R}})$. Then $\sup(\{f(x) \mid x \in X\})$ and $\inf(\{f(x) \mid x \in X\})$ exist.

PROOF: by (i) of the proposition $(\{f(x) \mid x \in X\}, d_{\mathbb{R}})$ is a weakly compact subspace of $(\mathbb{R},d_{\mathbb{R}})$. By (iii) $\sup(\{f(x) \mid x \in X\})$ and $\inf(\{f(x) \mid x \in X\})$ exist •

1.2.1 for a definition of 'compact' we are in search of an elegant stronger notion than 'weakly compact'. The following theorem suggests that 'fanlike' is a candidate.

THEOREM: every fanlike topological space is weakly compact.

PROOF: let $(X,\mathcal{T})$ be a fanlike topological space. Let $\mathcal{U}$ be an open cover of $(X,\mathcal{T})$. It suffices to prove that there is a finite sequence $U_0,\ldots,U_K$ of elements of $\mathcal{U}$ such that $\{U_i \mid i\le K\}$ is a cover of $(X,\mathcal{T})$. By remark 1.1.5 (i), without loss of generality X is a fan, say σ. By our remarks in 1.0.3 and lemma 1.0.4 we have that (σ,d_ω) refines $(\sigma,\mathcal{T})$. So we find:

$(\star)$ $\forall\alpha\in\sigma\ \exists n\in\mathbb{N}\ \exists U\in\mathcal{U}\ [\,\sigma\cap\overline{\alpha}(n)\subseteq U\,]$

By the fan theorem **FT** there is an $N\in\mathbb{N}$ such that for all a in $\overline{\overline{\sigma}}(N)$: $\exists U\in\mathcal{U}\,[\,\sigma\cap a\subseteq U\,]$. But $\overline{\overline{\sigma}}(N)$ is finite, since σ is a fan. Let $M\in\mathbb{N}$ be the number of elements of $\overline{\overline{\sigma}}(N)$, then we can find a finite sequence $U_0,\ldots,U_{M-1}$ of elements of $\mathcal{U}$ such that $\{U_i \mid i<M\}$ is a cover of $(\sigma,\mathcal{T})$ •

EXAMPLE: $(\{0\}\cup\{2^{-n}\mid n\in\mathbb{N}\},d_{\mathbb{R}})$ is a weakly compact space which is NOT fanlike.

Trivially the continuous image of a fanlike space is again fanlike. Now let f be a continuous function from a fanlike space $(X,\mathcal{T})$ to $(\mathbb{R},d_{\mathbb{R}})$. Then by the above theorem and corollary 1.2.0 we have that $\sup(\{f(x)\mid x\in X\})$ and $\inf(\{f(x)\mid x\in X\})$ exist. By the above theorem and proposition 1.2.0 (ii) a continuous function from a fanlike metric space to another metric space is uniformly continuous. This parallels the first three classical results mentioned in 1.2.0. But the next example shows that we cannot parallel the fourth of these classical results.

EXAMPLE: $([0,1]_3,d_{\mathbb{R}})$ is a fanlike metric space which is NOT complete (see remark 0.2.1).

Still $([0,1]_3,d_{\mathbb{R}})$ is the continuous image of (σ_2,d_ω), which is a fanlike complete metric space.

The theorem and these remarks show that 'fanlike' might be a good intuitionistic alternative to the classical notion 'compact'. We will however add the condition 'Hausdorff', see definition 1.2.2 below. With this extra condition we can prove that every compact space is an apartness space (theorem 1.2.4).

1.2.2* in this subsection we define two more concepts, called 'compact' and 'strongly compact'. We end up with four intuitionistic concepts which in increasing order of strength are: 'weakly compact', 'fanlike', 'compact' and 'strongly compact'.

DEFINITION: let $(X,\mathcal{T})$ be a topological space. Then $(X,\mathcal{T})$ is *compact* iff $(X,\mathcal{T})$ is fanlike and Hausdorff. $(X,\mathcal{T})$ is called *strongly compact* iff $(X,\mathcal{T})$ is compact and topologically complete.

REMARK: trivially a strongly compact space is compact, and a compact space is fanlike. By theorem 1.2.1 a fanlike space is weakly compact. It is perhaps interesting to note that our notion 'strongly compact' corresponds to the notion of 'compact' in Bishop's school and also to the notion of 'DFTK-space' in [Freudenthal37].

EXAMPLE: $([0,1], d_{\mathbb{R}})$ is a strongly compact space.

We will see that every compact space is a metrizable apartness space (theorem 1.2.4 and theorem 2.1.5).

1.2.3* observe that if $(X,\mathcal{T})$ is strongly compact, and d is a metric on X metrizing $(X,\mathcal{T})$, then (X,d) is complete. This because proposition 1.2.0 (ii) implies that a homeomorphism between fanlike metric spaces is uniformly continuous, and a uniformly continuous function preserves Cauchy-sequences (see 0.4.2). So a compact metric space is complete iff it is topologically complete iff it is strongly compact.

EXAMPLE: $([0,1]_3, d_{\mathbb{R}})$ is a compact space which is NOT strongly compact.

EXAMPLE: the halfline topology restricted to $[0,1]$ gives an example of a fanlike space which is NOT compact, since it is not Hausdorff.

Knowing classical topology, an interesting question is whether each first-separable weakly compact Hausdorff space is metrizable. We will prove instead, less generally, that each compact space is metrizable, see theorem 2.1.5.

1.2.4 the following theorem is fundamental to our analysis of compact spaces.

THEOREM: every compact space is an apartness space.

PROOF: it suffices to prove that every Hausdorff topological fan is an apartness space. let $(\sigma,\mathcal{T})$ be a Hausdorff topological fan. By 1.0.3 we have that $(\sigma,\#_{\mathcal{T}})$ refines $(\sigma,\mathcal{T})$. It suffices therefore to show that $(\sigma,\mathcal{T})$ refines $(\sigma,\#_{\mathcal{T}})$. Let U be open in $(\sigma,\#_{\mathcal{T}})$, and let α be in U.

$\boxed{\text{claim}}$ there is a V in $\mathcal{T}$ such that $\alpha \in V$ and $V \subseteq U$.

$\boxed{\text{proof}}$ let β be in σ. Then by definition of the apartness topology we have: $\beta \in U$ or $\beta \#_{\mathcal{T}} \alpha$. The latter case implies, by definition of Hausdorff, that there are V and W in $\mathcal{T}$ such that $\alpha \in V$ and $\beta \in W$ and $V \cap W = \emptyset$. We find:

$$(\star) \quad \forall \beta \in \sigma\ \exists s \in \{0,1\}\ [\,(s=0 \wedge \beta \in U) \vee (s=1 \wedge \beta \#_{\mathcal{T}} \alpha)\,]$$

By the fan theorem **FT** there is an $N \in \mathbb{N}$ and a function h from $\overline{\overline{\sigma}}(N)$ to $\{0,1\}$ such that for all β in σ: $h(\overline{\beta}(N))=0$ implies $\beta \in U$ and $h(\overline{\beta}(N))=1$ implies $\beta \#_{\mathcal{T}} \alpha$.

$\boxed{\text{case 1}}$ $h(a)=0$ for all a in $\overline{\overline{\sigma}}(N)$.
Then $U=\sigma$ so trivially the claim is true.

$\boxed{\text{case 2}}$ there is an a in $\overline{\overline{\sigma}}(N)$ such that $h(a)=1$.
Then $\tau = \{\beta \in \sigma \mid h(\overline{\beta}(N))=1\}$ is a subfan of σ. Let β be in τ, then $\beta \#_{\mathcal{T}} \alpha$. Since $(\sigma, \mathcal{T})$ is Hausdorff there are V and W in $\mathcal{T}$ such that $\alpha \in V$ and $\beta \in W$ and $V \cap W = \emptyset$. By our remarks in 1.0.3 and lemma 1.0.4 we have that (σ, d_ω) refines $(\sigma, \mathcal{T})$. So we find:

$$(\star\star) \quad \forall \beta \in \tau\ \exists n \in \mathbb{N}\ \exists V, W \in \mathcal{T}\ [\,\alpha \in V \wedge \tau \cap \overline{\beta}(n) \subseteq W \wedge V \cap W = \emptyset\,]$$

So by the fan theorem **FT** there is an $M \in \mathbb{N}$ such that for all a in $\overline{\overline{\tau}}(M)$: there are V, W in $\mathcal{T}$ with the property that $\alpha \in V$ and $\tau \cap a \subseteq W$ and $V \cap W = \emptyset$. But $\overline{\overline{\tau}}(M)$ is finite, so there is a finite sequence $(V_0, W_0), \ldots, (V_K, W_K)$ of pairs of elements of $\mathcal{T}$ such that for all $i \leq K$: $\alpha \in V_i$ and $V_i \cap W_i = \emptyset$, and moreover $\forall \beta \in \tau\ \exists i \leq K\ [\beta \in W_i]$. Put $V = \bigcap_{i \leq K} V_i$. Then V is in $\mathcal{T}$, α is in V, and $V \subseteq U$ since for all β in V we must have $h(\overline{\beta}(N))=0$ $\circ$

Since α is arbitrary, the claim together with O_2 yields that U is in $\mathcal{T}$ $\bullet$

COROLLARY: every function from a compact space to another topological space is continuous (see thm. 1.1.0).

REMARK: the theorem shows that every compact space coincides with an apartness fan. Conversely every apartness fan is compact. We will prove this in chapter two, by showing that every apartness fan is Hausdorff (even metrizable), see section 2.1.

1.3 STRONGLY SUBLOCATED SUBSPACES

1.3.0* DEFINITION: let $(X,\mathcal{T})$ be a topological space. Let A be a subset of $(X,\mathcal{T})$. Then A is called *separable* in $(X,\mathcal{T})$ iff there is a sequence $(a_n)_{n\in\mathbb{N}}$ of elements of A which is $\mathcal{T}$-dense in A, that is: $\forall a\in A\ \forall U\ni a\ \exists n\in\mathbb{N}\ [\,a_n\in U\,]$.

Let A be a separable subset of $(X,\mathcal{T})$. Put $\mathcal{T}_A = \mathcal{T}\cap A = \{U\cap A \mid U\in\mathcal{T}\}$. Then $(A,\mathcal{T}_A)$ is a topological space satisfying O_1 through O_5 along with O_7. We call $(A,\mathcal{T}_A)$ a *subspace* of $(X,\mathcal{T})$, and we say that $\mathcal{T}_A$ is the *subspace topology* on A (relative to $(X,\mathcal{T})$).

REMARK: careful with apartness spaces! For let $(X,\#)$ be an apartness space, and let $\mathcal{T}=\mathcal{T}_\#$ be the $\#$-topology on X. Let A be a subset of $(X,\#)$, then $(A,\mathcal{T}_A)$ need not coincide with $(A,\#)$, the topological space arising from restricting $\#$ to A. As an example take $A=\{\alpha\in\sigma_{2\mathrm{mon}}\,|\,\alpha=\underline{0}\vee\alpha\,\#_\omega\underline{0}\}$ and $(X,\#)=(\sigma_{2\mathrm{mon}},\#_\omega)$. Then $(A,\mathcal{T}_A)$ coincides with $(\{0\}\cup\{2^{-n}\,|\,n\in\mathbb{N}\},d_{\mathbb{R}})$ whereas $(A,\#_\omega)$ coincides with $(\mathbb{N},d_{\mathbb{R}})$. Of course by 1.0.3 we have that if $(A,\mathcal{T}_A)$ is a subspace of $(X,\#)$, then $(A,\#)$ refines $(A,\mathcal{T}_A)$.

1.3.1* LEMMA: let $(A,\mathcal{T}_A)$ be a subspace of $(X,\#)$ such that A is open in $(X,\#)$. Then $(A,\mathcal{T}_A)$ coincides with $(A,\#)$.

PROOF: by 1.0.3 $(A,\#)$ refines $(A,\mathcal{T}_A)$, so it suffices to check that $(A,\mathcal{T}_A)$ refines $(A,\#)$. Let U be open in $(A,\#)$. Let $x\in U$ and let y be in X. Since x is in A we can decide: $y\in A$ or $x\#y$. If $y\in A$ we can decide: $y\in U$ or $x\#y$. Therefore we can always decide: $y\in U$ or $x\#y$, meaning U is open in $(X,\#)$. Then a fortiori U is open in $(A,\mathcal{T}_A)$ •

1.3.2* DEFINITION: let A be a subset of $(X,\mathcal{T})$. We say that A is *dense* in $(X,\mathcal{T})$ iff there is a sequence $(a_n)_{n\in\mathbb{N}}$ of elements of A such that for all inhabited U in $\mathcal{T}$, there is an $n\in\mathbb{N}$ such that a_n is in $U\cap A$. Then A is separable in $(X,\mathcal{T})$, and by extension we call the subspace $(A,\mathcal{T}_A)$ *dense* in $(X,\mathcal{T})$.

1.3.3* DEFINITION: let $(A,\mathcal{T}_A)$ be a subspace of $(X,\mathcal{T})$. We say that $(A,\mathcal{T}_A)$ is (i) *sublocated* (ii) *strongly sublocated* in $(X,\mathcal{T})$ iff

(i) $\forall x\in X\ \forall U\ni x\ [\,\exists a\in A\,[a\in U]\vee\exists V\ni x\,[V\cap A=\emptyset]\,]$

(ii) $\forall x\in X\ \exists y\in A\ [\,x\#y\rightarrow\exists U\ni x\,[U\cap A=\emptyset]\,]$.

REMARK:

(i) if $(A,\mathcal{T}_A)$ is strongly sublocated in $(X,\mathcal{T})$, then $(A,\mathcal{T}_A)$ is sublocated in $(X,\mathcal{T})$. For suppose $(A,\mathcal{T}_A)$ is strongly sublocated in $(X,\mathcal{T})$. Let x in X, and $U\ni x$ open in $(X,\mathcal{T})$. Find y in A such that $x\#y$ implies $\exists V\ni x\,[V\cap A=\emptyset]$. Since $(X,\mathcal{T})$ is effective we can decide: $y\in U$ or $x\#y$. On the other hand, take $A=\{2^{-n}\,|\,n\in\mathbb{N}\}$ and $(X,\mathcal{T})=(\{0\}\cup\{2^{-n}\,|\,n\in\mathbb{N}\},d_{\mathbb{R}})$. Then $(A,\mathcal{T}_A)$ is sublocated in $(X,\mathcal{T})$, but NOT strongly sublocated in $(X,\mathcal{T})$.

(ii) our notion 'sublocated in' occurs in [Troelstra&vanDalen88, def.7.3.2] where it is called 'topologically located in'. We will use the expression 'topologically located in' to indicate the following: $(A,\mathcal{T}_A)$ is *topologically located* in $(X,\mathcal{T})$ iff there is a metric d on X such that $(X,\mathcal{T})$ coincides with (X,d), and (A,d) is located in (X,d) (see 1.1.2, and definitions 3.2.0 and 3.2.2).

(iii) we believe that 'strongly sublocated in' is a more useful notion than 'sublocated in'. We also think that these notions are of interest first and foremost in a metric context. We will discuss them again in chapter three. Notice that classically 'sublocated in' would be an empty condition, whereas being 'strongly sublocated in' would correspond to being a 'closed subset of'.

(iv) for apartness spaces a nice alternative phrasing of 'strongly sublocated in' is given in the following lemma:

LEMMA: a subspace $(A,\mathcal{T}_A)$ of an apartness space $(X,\#)$ is strongly sublocated in $(X,\#)$ iff $\forall x\in X\ \exists y\in A\ [\,x\#y\rightarrow\forall a\in A\,[x\#a]\,]$.

PROOF: suppose $\forall x\in X\ \exists y\in A\ [\,x\#y\ \rightarrow\ \forall a\in A\,[x\#a]\,]$. Then we hold that $X\setminus\!\setminus_A=\{z\in X\,|\,\forall a\in A\,[z\#a]\}$ is open in $(X,\#)$. For let z in $X\setminus\!\setminus_A$, and x in X. Determine y in A such that $x\#y$ implies $\forall a\in A\,[x\#a]\,]$. Now $z\#y$ so we can decide: $x\#z$ or $x\#y$, the latter implying that x is in $X\setminus\!\setminus_A$. Trivially $X\setminus\!\setminus_A\cap A=\emptyset$. From this it is easy to conclude that $(A,\mathcal{T}_A)$ is strongly sublocated in $(X,\#)$. The implication the other way round is trivial •

COROLLARY: let $(A,\mathcal{T}_A)$ be strongly sublocated in $(X,\#)$, then $(A,\mathcal{T}_A)$ coincides with $(A,\#)$.

PROOF: by 1.0.3 $(A,\#)$ refines $(A,\mathcal{T}_A)$, so it suffices to check that $(A,\mathcal{T}_A)$ refines $(A,\#)$. Let V be open in $(A,\#)$. Put $U=\{x\in X\,|\,\exists y\in V\ [\,x\#y\rightarrow\forall b\in A\,[x\#b]\,]\}$. We show that U is open in $(X,\#)$. Let x be in U, and x_A in V such that $x\#x_A$ implies $\forall b\in A\,[\,x\#b\,]$. Let z be in X. Determine z_A in A such that $z\#z_A$ implies $\forall b\in A\,[\,z\#b\,]$. Since $V\ni x_A$ is open in $(A,\#)$ we can decide on one of the following cases:

$\boxed{\text{case 1}}$ z_A is in V.
then z is in U.

$\boxed{\text{case 2}}$ $z_A \# x_A$.
then $x \# z_A$ (!) and so:

$\boxed{\text{case 2.1}}$ $z \# x$.

$\boxed{\text{case 2.2}}$ $z \# z_A$.
but then z is in U since: x_A is in V and $z \# x_A$ implies $\forall b \in A\, [z \# b]$.

So for all x in U and all z in X: $z \in U$ or $z \# x$, meaning U is open in $(X, \#)$. It is easy to see, on the other hand, that $V = U \cap A$, and therefore V is open in $(A, \mathcal{T}_A)$ •

1.3.4* we wish to show that 'strongly sublocated in' behaves transitively.

PROPOSITION: let $(B, \mathcal{T}_B)$ be strongly sublocated in $(A, \mathcal{T}_A)$, where $(A, \mathcal{T}_A)$ is strongly sublocated in $(X, \mathcal{T})$. Then $(B, \mathcal{T}_B)$ is strongly sublocated in $(X, \mathcal{T})$.

PROOF: let x be in X. Determine a, b in A, B respectively such that $x \# a$ implies $\exists U \ni x\, [U \cap A = \emptyset]$ and $a \# b$ implies $\exists V \ni a, V \in \mathcal{T}_A\, [V \cap B = \emptyset]$, which in turn implies $\exists V \ni a, V \in \mathcal{T}\, [V \cap B = \emptyset]$. Suppose $x \# b$. Then $x \# a$ which implies $\exists U \ni x\, [U \cap B = \emptyset]$, or $a \# b$. Suppose $a \# b$. Then there is $V \ni a$ with $V \cap B = \emptyset$. Since $(X, \mathcal{T})$ is effective we can decide $x \in V$ or $x \# a$, and in both cases we find a $U \ni x$ with $U \cap A = \emptyset$. Since x is arbitrary we find: $\forall x \in X\, \exists a \in A\, [x \# a \rightarrow \exists U \ni x\, [U \cap A = \emptyset]]$ •

EXAMPLE: we give a Brouwerian counterexample to the statement 'if $(B, \mathcal{T}_B)$ is sublocated in $(A, \mathcal{T}_A)$, where $(A, \mathcal{T}_A)$ is sublocated in $(X, \mathcal{T})$, then $(B, \mathcal{T}_B)$ is sublocated in $(X, \mathcal{T})$'. Let $E = \overline{\{2^{-n} \mid n \in \mathbb{N}\}}$ (w.r.t. $d_{\mathbb{R}}$), let $B = \{1\} \cup \{e \in E \mid \exists n \in \mathbb{N}\, [n = k_{99}]\}$, put $A = B \cup \{3^{-n} \mid n \in \mathbb{N}\}$ and $X = A \cup \{0\}$, and let $d = d_{\mathbb{R}}$. Then (B, d) is sublocated in (A, d) and (A, d) is sublocated in (X, d), but if (B, d) is sublocated in (X, d), then we can decide: $\exists n \in \mathbb{N}\, [n = k_{99}]$ or $\forall n \in \mathbb{N}\, [n < k_{99}]$.

1.3.5* DEFINITION: let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces. A function f from $(X, \mathcal{T})$ to $(Y, \mathcal{T}')$ is called an *embedding* of $(X, \mathcal{T})$ in $(Y, \mathcal{T}')$ iff f is a homeomorphism from $(X, \mathcal{T})$ to $(f(X), \mathcal{T}'_{f(X)})$.

1.4 LOCAL PROPERTIES

1.4.0* this section is devoted to an analysis of what we mean with a 'local' property, such as 'locally compact' or 'locally connected'. We will follow the classical approach. Let us call a subset W of $(X, \mathcal{T})$ a *neighborhood* of $x \in X$ iff there is an open $U \ni x$ such that $U \subseteq W$. A collection $\mathcal{W}$ of neighborhoods is called a *cover* of $(X, \mathcal{T})$ iff for each x in $(X, \mathcal{T})$ there is a W in $\mathcal{W}$ and an open $U \ni x$ such that $U \subseteq W$. A cover $\mathcal{W}$ is said to *refine* a cover $\mathcal{V}$ iff for all W in $\mathcal{W}$ there is a V in $\mathcal{V}$ with $W \subseteq V$. Now suppose we have a topological concept C. Frequently we would like a space $(X, \mathcal{T})$ to be 'locally C'. We believe that this should indicate the following:

L_0. for all x in X, for all $U \ni x$: there is a neighborhood $W \subseteq U$ of x such that $(W, \mathcal{T}_W)$ is a topological space which is C.

To ensure a certain manageability of the 'local' property on the space $(X, \mathcal{T})$ as a whole, one is glad whenever:

L_1. $(X, \mathcal{T})$ is spreadlike.

If $(X, \mathcal{T})$ is a first-separable space satisfying L_0 and L_1, then by proposition 1.1.6 every open cover $\mathcal{U}$ of $(X, \mathcal{T})$ has an enumerable refinement $\{U_n | n \in \mathbb{N}\}$ such that each U_n is contained in a subspace $(W, \mathcal{T}_W)$ of $(X, \mathcal{T})$ which is C.

We propose to adopt the terminology '1-locally C' for spaces satisfying both L_0 and L_1.

1.4.1 we apply this guideline to 'locally compact'.

DEFINITION: a topological space $(X, \mathcal{T})$ is called *locally compact* iff it satisfies L_0 with respect to 'compact'. It is called *1-locally compact* iff in addition it is spreadlike.

LEMMA: let $(X, \mathcal{T})$ be a topological space. Suppose for all x in X there is a separable neighborhood $W \ni x$ such that the subspace $(W, \mathcal{T}_W)$ coincides with an apartness space $(Y, \#_Y)$. Then $(X, \mathcal{T})$ coincides identically with $(X, \#)$.

PROOF: we know by our analysis in 1.0.3 that $(X, \#)$ refines $(X, \mathcal{T})$, so we show only that $(X, \mathcal{T})$ refines $(X, \#)$. Let U be open in $(X, \#)$, and let x be in U. Determine a separable neighborhood $W \ni x$ such that $(W, \mathcal{T}_W)$ coincides with an apartness space $(Y, \#_Y)$. Determine a $V \ni x$ open in $(X, \mathcal{T})$ such that $V \subseteq W$. Then $(W, \mathcal{T}_W)$ coincides identically with $(W, \#)$ (lemma 1.1.1 (ii)). Therefore $U \cap W$ is open in $(W, \mathcal{T}_W)$, meaning there is a U' open in $(X, \mathcal{T})$ such that $U \cap W = U' \cap W$. Then $V \cap U'$ is open in $(X, \mathcal{T})$

and $V\cap U'=V\cap U\subseteq U$ and x is in $V\cap U'$. O_2 together with the arbitrariness of x now implies that U is open in $(X,\mathcal{T})$ •

COROLLARY:

(i) every locally compact space is an apartness space (see theorem 1.2.4).

(ii) every function from a locally compact space to another topological space is continuous (see theorem 1.1.0)

1.4.2 the previous corollary generalizes theorem 0.4.1. However in its proof we have used the fan theorem **FT**, whereas we used only the continuity principle **CP** for the proof of theorem 0.4.1.

1.4.3* DEFINITION: a space $(X,\mathcal{T})$ is called *(1-)locally strongly compact* iff it satisfies the corresponding definitions above with 'compact' replaced by 'strongly compact'. It is called *(1-)locally weakly compact* iff it satisfies the corresponding definitions above with 'compact' replaced by 'weakly compact'.

1.4.4* EXAMPLE: we define the *one-point weak compactification* $(\sigma_{\omega_{-1}},\mathcal{T}_{-1})$ of $(\sigma_\omega,\#_\omega)$ as follows. Put $\sigma_{\omega_{-1}}=\sigma_\omega\cup\{-1\}$. Let U be a subset of $\sigma_{\omega_{-1}}$, then U is in $\mathcal{T}_{-1}$ iff $U\cap\sigma_\omega$ is open in $(\sigma_\omega,\#_\omega)$ and: $-1\in U$ implies that there is a weakly compact subspace $(A,\mathcal{T}_A)$ of $(\sigma_\omega,\#_\omega)$ such that $\sigma_{\omega_{-1}}=U\cup A$. One verifies that $(\sigma_{\omega_{-1}},\mathcal{T}_{-1})$ satisfies O_1 through O_5 as well as O_7. Trivially $(\sigma_{\omega_{-1}},\mathcal{T}_{-1})$ is T_1 and weakly compact, but NOT locally weakly compact.

REMARK: for an arbitrary topological space $(X,\mathcal{T})$ we can define the 'one-point weakly compact extension' in a completely similar way.

1.4.5 we study $(\sigma_{\omega_{-1}},\mathcal{T}_{-1})$ with respect to O_6. Is $(\sigma_{\omega_{-1}},\mathcal{T}_{-1})$ first-separable, that is: does it have an enumerable basis?

Notice that a weakly compact subspace $(A,\mathcal{T}_A)$ of $(\sigma_\omega,\#_\omega)$ is contained in a subfan of σ_ω. For $\{(\sigma_\omega\cap a)\cap A\mid a\in\overline{\overline{\sigma}}_\omega(1)\}$ is an open cover of $(A,\mathcal{T}_A)$. Therefore there are $a_0,\ldots a_n$ in $\overline{\overline{\sigma}}_\omega(1)$ such that A is contained in $\cup_{i\leq n}(\sigma_\omega\cap a_i)$. Then we turn to $\{(\sigma_\omega\cap a)\cap A\mid a\in\overline{\overline{\sigma}}_\omega(2)\}$, and so on. With a little care we construct a subfan τ of σ_ω such that A is contained in τ.

Now suppose there is a sequence $(V_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{T}_{-1}$ containing -1 such that for all U in $\mathcal{T}_{-1}$ which contain -1 there is an $n\in\mathbb{N}$ with $x\in V_n\subseteq U$. By $\mathbf{AC}_{01}$ we obtain a sequence $(\tau_n)_{n\in\mathbb{N}}$ of subfans of σ_ω such that for each $n\in\mathbb{N}$: $\sigma_{\omega_{-1}}=V_n\cup\tau_n$. Define an α in σ_ω as follows. For $n\in\mathbb{N}$ put $\alpha(n)=\max(\{\beta(n)\,|\,\beta\in\tau_n\})+1$. Then $\{\alpha\}$ is a fan, therefore $U=\{\beta\in\sigma_\omega\,|\,\beta\#_\omega\alpha\}\cup\{-1\}$ is an open neighborhood of -1. But for all $n\in\mathbb{N}$: α is in U_n. Contradiction.

But if $(\sigma_{\omega_{-1}},\mathcal{T}_{-1})$ is first-separable, then such a sequence $(V_n)_{n\in\mathbb{N}}$ must exist, by the following reasoning. Suppose there is a sequence $(U_n)_{n\in\mathbb{N}}$ of inhabited elements of $\mathcal{T}_{-1}$ such that for all $x\in U\in\mathcal{T}_{-1}$ there is an $n\in\mathbb{N}$ with $x\in U_n\subseteq U$. Notice that every subfan τ of σ_ω in fact codes a U in $\mathcal{T}_{-1}$ which contains -1. Remember that there is a spread σ_{fan} which contains precisely all (encodings of) subfans of σ_ω (see 0.0.4). Combine this to find:

$$(\star)\quad \forall\tau\in\sigma_{\text{fan}}\ \exists n\in\mathbb{N}\ [\,U_n\cup\tau=\sigma_{\omega_{-1}}\,]$$

By $\mathbf{AC}_{10}$ there is a spread-function γ from σ_{fan} to $\mathbb{N}$ realizing $(\star)$. Put $A=\{n\in\mathbb{N}\mid\exists a\in\overline{\overline{\sigma}}_{\text{fan}}\,[\gamma(a)=n+1]\}$. Clearly A is enumerable, and clearly : $(U_n)_{n\in A}$ is a sequence of open sets containing -1 such that for all open U which contain -1: there is an $n\in A$ such that $-1\in U_n\subseteq U$.

This shows that $(\sigma_{\omega_{-1}},\mathcal{T}_{-1})$ is NOT first-separable (so NOT O_6).

REMARK: a very similar but *enumerable* example of a topological space which is NOT first-separable is given in [Urysohn25a, Anhang II].

1.4.6* DEFINITION: we say that a topological space $(X,\mathcal{T})$ is *(1-)locally connected* iff $(X,\mathcal{T})$ satisfies L_0 (and L_1) with respect to 'connected'. The same, mutatis mutandis, for *(1-)locally pathwise connected* and *(1-)locally arcwise connected*.

CHAPTER TWO

APARTNESS TOPOLOGY

abstract

We study the general setting of a Σ^1_0-apartness spread $(\sigma, \#)$. We find examples of a weakly compact $(\sigma, \#)$ which is T_1 but NOT Hausdorff, and of a $(\sigma, \#)$ which is Hausdorff but NOT regular. We also see that $(\sigma, \#)$ is normal if $\#$ is the natural apartness $\#_\omega$. We prove that every apartness fan is metrizable. Therefore a topological space $(X, \mathcal{T})$ is compact iff $(X, \mathcal{T})$ coincides with an apartness fan. Thus every compact space is metrizable. We also obtain that $([0,1]_3, d_{\mathbb{R}})$ is pathwise connected but NOT arcwise connected. A topological space $(X, \mathcal{T})$ is 1-locally compact iff it admits an enumerable cover of compact neighborhoods. Every 1-locally compact space has a one-point compact extension and is metrizable. Introduction of sigma-compact spaces. Every sigma-compact space is weakly metrizable; NOT every sigma-compact space is metrizable. Introduction of *star-finitary* spaces, which form a generalization of 1-locally compact spaces. Every star-finitary space is metrizable.

2.0 APARTNESS SPREADS

2.0.0* in this chapter we primarily study apartness spreads $(\sigma,\#)$, and especially the question whether $(\sigma,\#)$ can be metrized. Remember convention 1.0.7 that every apartness spread $(\sigma,\#)$ is tacitly assumed to be a Σ^1_0-apartness spread. There are some connections with [Freudenthal36] and [Troelstra66]. Many topological spaces occurring in these references may be viewed as Σ^1_0-apartness spreads. However since their topology is obtained in a different way, comparison is difficult. There are also connections with the nice paper [Urysohn25a], especially in 2.0.4.

2.0.1* we introduce some notations.

DEFINITION: let σ be a spread, and let a be in $\overline{\overline{\sigma}}$. We write $\sigma\cap a$ for the subspread $\{\alpha\in\sigma \mid \alpha(lg(a))=a\}$ of σ. Let A be a subset of $\overline{\overline{\sigma}}$. We write σ_A for $\bigcup_{a\in A}\sigma\cap a$. Finally we write $\overline{\overline{\sigma}}_A(n)$ for $\{\overline{\alpha}(n)\,|\,\alpha\in\sigma_A\}$ and $\overline{\overline{\sigma}}_A$ for $\bigcup_{n\in\mathbb{N}}\overline{\overline{\sigma}}_A(n)$.

2.0.2 recall (1.0.5) that the natural apartness $\#_\omega$ on σ_ω is induced by a decidable subset $\approx$ of $\overline{\overline{\sigma}}_\omega\times\overline{\overline{\sigma}}_\omega$. We wish to show that such is the fate of any Σ^1_0-apartness on a spread σ.

DEFINITION: let σ be a spread, and let $\approx$ be a subset of $\overline{\overline{\sigma}}\times\overline{\overline{\sigma}}$, with complement $\not\approx$. Then $\approx$ is called a *touch-relation* on $\overline{\overline{\sigma}}$ iff

(i) $\forall a,b,c\in\overline{\overline{\sigma}}\;[\,a\approx a\wedge(b\sqsubseteq c\rightarrow(a\approx c\vee b\not\approx a))\,]$

(ii) $\forall a,b\in\overline{\overline{\sigma}}\;[\,a\not\approx b\rightarrow\forall\alpha\in\sigma\cap a\,\forall\beta\in\sigma\cap b\,\forall\gamma\in\sigma\,\exists n\in\mathbb{N}\,[\overline{\gamma}(n)\not\approx\overline{\alpha}(n)\vee\overline{\gamma}(n)\not\approx\overline{\beta}(n)]\,]$

We do not wish to deprive the reader of the amusing exercise to show that (i) expresses all of the following: $\approx$ is a decidable symmetric reflexive subset of $\overline{\overline{\sigma}}\times\overline{\overline{\sigma}}$ such that $\not\approx$ is monotone ($a\not\approx b$ implies $a\not\approx b\star c$). Then (ii) expresses that $\approx$ induces a Σ^1_0-apartness $\#$ on σ by putting $\alpha\#\beta$ iff $\exists n\in\mathbb{N}\;[\,\overline{\alpha}(n)\not\approx\overline{\beta}(n)\,]$.

EXAMPLE: an important example is of course the touch-relation $\approx_{\mathbb{R}}$ defined in 0.2.0.

PROPOSITION: let $\#$ be a Σ^1_0-apartness on a spread σ. Then there is a touch-relation $\approx$ on $\overline{\overline{\sigma}}$ such that for all α, β in σ: $\alpha\#\beta$ iff $\exists n\in\mathbb{N}\;[\,\overline{\alpha}(n)\not\approx\overline{\beta}(n)\,]$.

PROOF: by definition of 'Σ^1_0-apartness' (1.0.5) we have:

$(\star)$ $\quad \forall(\alpha,\beta)\in\sigma\times\sigma\ \exists\delta\in\sigma_{2\mathrm{mon}}\ [\,\alpha\#\beta \leftrightarrows \exists n\in\mathbb{N}\,[\delta(n)=1]\,]$

By $\mathbf{AC}_{11}$ there is a spread-function γ from $\sigma\times\sigma$ to $\sigma_{2\mathrm{mon}}$ realizing $(\star)$. For $a\,,b$ in $\overline{\overline{\sigma}}$ put $a\bowtie b=\lessdot\lessdot a_0,b_0\gtrdot,\ldots,\lessdot a_{s-1},b_{s-1}\gtrdot\gtrdot$, where $s=\min(lg(a),lg(b))$. Then $a\bowtie b$ is in $\overline{\overline{\sigma\times\sigma}}$. We put $\approx\,=\{(a,b)\in\overline{\overline{\sigma}}\times\overline{\overline{\sigma}}\mid\gamma_{[lg(a\star b)]}(a\bowtie b)\le 1\}$. It is easy to see that $\approx$ is as promised $\bullet$

2.0.3 in order to get more feeling with the subject we study some examples of Σ^1_0-apartness spreads. We already know that any apartness space $(X,\#)$ is T_1 (corollary 1.1.7).

EXAMPLE: we give an example of a weakly compact $(\sigma,\#)$ which is T_1 and NOT Hausdorff (T_2). We describe σ by specifying $\overline{\overline{\sigma}}$. Put

(i) $A=\{\,\underline{0}(n)\star\lessdot p_n^{m+2}\gtrdot\star\underline{0}(s)\mid n,m,s\in\mathbb{N}\,\}$

(ii) $B=\{\,\lessdot p_n\gtrdot\star a\mid n\in\mathbb{N},a\in\overline{\overline{\sigma}}_{2\mathrm{mon}}\}$

(iii) $C=\{\,\underline{1}(n)\star\lessdot p_n\gtrdot\star\underline{1}(m)\mid n,m\in\mathbb{N}\,\}$

where $(p_n)_{n\in\mathbb{N}}$ is the standard enumeration of the prime numbers. Next, let $\overline{\overline{\sigma}}=\{a\in\mathbb{N}\,|\,\exists b\in A\cup B\cup C\,[a\sqsubseteq b]\}$. Notice that $\overline{\overline{\sigma}}$ is decidable in $\mathbb{N}$. We specify a touch-relation $\approx$ on $\overline{\overline{\sigma}}$ by lettin $\approx$ be the smallest subset of $\overline{\overline{\sigma}}\times\overline{\overline{\sigma}}$ satisfying:

(i) $\forall n,m\in\mathbb{N}\ [\,\underline{1}(n)\star\lessdot p_n\gtrdot\star\underline{1}(m)\approx\lessdot p_n\gtrdot\star\underline{0}(m)\,]$

(ii) $\forall n,m,s\in\mathbb{N}\ [\,\underline{0}(n)\star\lessdot p_n^{m+2}\gtrdot\star\underline{0}(s)\approx\lessdot p_n\gtrdot\star\underline{0}(m)\star\underline{1}(s)\,]$

(iii) $\forall a,b,c,d\in\overline{\overline{\sigma}}\ [\,a\approx a\wedge((a\sqsubseteq c\wedge b\sqsubseteq d\wedge c\approx d)\to a\approx b)\,]$

Let us clarify this example a little bit. Identify, popularly speaking, p_n^{m+2} with the $\equiv$-equivalent sequences $\underline{0}(n)\star\lessdot p_n^{m+2}\gtrdot\star\underline{0}$ and $\lessdot p_n\gtrdot\star\underline{0}(m)\star\underline{1}$. Identify p_n with $p_n\star\underline{0}$. We have that $\underline{0}\#\underline{1}$ since already $\lessdot 0\gtrdot\not\approx\lessdot 1\gtrdot$. Furthermore we have, popularly speaking, that $\{p_n^{m+2}\,|\,n,m\in\mathbb{N}\}$ accumulates on $\underline{0}$. Also, for each $n\in\mathbb{N}$: $\{p_n^{m+2}\,|\,m\in\mathbb{N}\}$ accumulates $\sigma_{2\mathrm{mon}}$-like on p_n . Finally, $\{p_n\,|\,n\in\mathbb{N}\}$ accumulates $\sigma_{2\mathrm{mon}}$-like on $\underline{1}$. To see that $(\sigma,\#)$ is NOT T_2, let $U\ni\underline{0}$, $V\ni\underline{1}$ be open in $(\sigma,\#)$. Since (σ,d_ω) refines $(\sigma,\#)$, there is an $N\in\mathbb{N}$ such that for all $n\in\mathbb{N}$, $n>N$: $p_n\in V$ and $p_n^{m+2}\in U$ for all $m\in\mathbb{N}$. So p_{N+1} is in V . Since (σ,d_ω) refines $(\sigma,\#)$, there is an $M\in\mathbb{N}$ such that for all $m\in\mathbb{N}$, $m>M$: p^m_{N+1} is in V . Therefore the intersection $U\cap V$ is inhabited. We leave it to the reader to verify that $(\sigma,\#)$ is weakly compact.

REMARK: notice that we have all but described a first enumerable space $(X,\mathcal{T})$ which is weakly compact, T_1 but NOT Hausdorff. For we can take $X=\{0\}\cup\{p_n^m\,|\,n,m\in\mathbb{N}\}$, and let $\mathcal{T}$ be generated by the countable basis $D\cup E\cup F\cup G$ where

(i) $D=\{\,\{p_n^{m+2}\}\mid n,m\in\mathbb{N}\,\}$

(ii) $E=\{\,\{p_n\}\cup\{p_n^{m+2+N}\,|\,m\in\mathbb{N}\}\mid n,N\in\mathbb{N}\,\}$

(iii) $F=\{\,\{0\}\cup\{p_{n+N}^{m+2}\,|\,n,m\in\mathbb{N}\}\mid N\in\mathbb{N}\,\}$

(iv) $G=\{\,\{1\}\cup\{p_{n+N}\,|\,n\in\mathbb{N}\}\cup\{p_{n+N}^{m+2}\,|\,n,m\in\mathbb{N}\}\mid N\in\mathbb{N}\,\}$

Classically $(\sigma,\#)$ would be the quotient topological space (of (σ,d_ω)) arising from the equivalence relation $\equiv$ induced by $\#$. One then has classically that $(\sigma,\#)$ and $(X,\mathcal{T})$ coincide. But of course intuitionistically $(\sigma,\#)$ and $(X,\mathcal{T})$ do NOT coincide. To be precise $(X,\mathcal{T})$ coincides with the subspace $(Y,\mathcal{T}_Y$ of $(\sigma,\#)$, where $Y=\{\alpha\in\sigma\mid(\alpha\#\underline{0}\vee\alpha\equiv\underline{0})\wedge(\alpha\#\underline{1}\vee\alpha\equiv\underline{1})\}$ and $\mathcal{T}_Y$ is the subspace topology. Our next example is similar:

2.0.4 EXAMPLE: we give an example of a $(\sigma,\#)$ which is Hausdorff (T_2) and NOT regular (T_3). We describe σ by specifying $\overline{\overline{\sigma}}$. Put

(i) $A=\{\,\overline{\underline{0}}(n)\star\lessdot p_n^{2m+2}\gtrdot\star\overline{\underline{0}}(s)\mid n,m,s\in\mathbb{N}\,\}$

(ii) $B=\{\,\lessdot p_n\gtrdot\star a\mid n\in\mathbb{N},a\in\overline{\overline{\sigma}}_{2\mathrm{mon}}\}$

(iii) $C=\{\,\overline{\underline{1}}(n)\star\lessdot p_n^{2m+3}\gtrdot\star\overline{\underline{1}}(s)\mid n,m,s\in\mathbb{N}\,\}$

where $(p_n)_{n\in\mathbb{N}}$ is the standard enumeration of the prime numbers. Next, let $\overline{\overline{\sigma}}=\{a\in\mathbb{N}\,|\,\exists b\in A\cup B\cup C\,[a\sqsubseteq b]\}$. Notice that $\overline{\overline{\sigma}}$ is decidable in $\mathbb{N}$. We specify a touch-relation $\approx$ on $\overline{\overline{\sigma}}$ by letting $\approx$ be the smallest subset of $\overline{\overline{\sigma}}\times\overline{\overline{\sigma}}$ satisfying:

(i) $\forall n,m,s\in\mathbb{N}\;[\,\overline{\underline{1}}(n)\star\lessdot p_n^{2m+3}\gtrdot\star\overline{\underline{1}}(s)\approx\lessdot p_n\gtrdot\star\overline{\underline{0}}(2m+1)\star\overline{\underline{1}}(s)\,]$

(ii) $\forall n,m,s\in\mathbb{N}\;[\,\overline{\underline{0}}(n)\star\lessdot p_n^{2m+2}\gtrdot\star\overline{\underline{0}}(s)\approx\lessdot p_n\gtrdot\star\overline{\underline{0}}(2m)\star\overline{\underline{1}}(s)\,]$

(iii) $\forall a,b,c,d\in\overline{\overline{\sigma}}\;[\,a\approx a\wedge((a\sqsubseteq c\wedge b\sqsubseteq d\wedge c\approx d)\rightarrow a\approx b)\,]$

We clarify this example a little bit: we have that $\underline{0}\#\underline{1}$ since already $\lessdot 0\gtrdot\not\approx\lessdot 1\gtrdot$. Furthermore we have, popularly speaking (see the previous example), that $\{p_n^{2m+2}\,|\,n,m\in\mathbb{N}\}$ accumulates $\sigma_{2\mathrm{mon}}$-like on $\underline{0}$, and $\{p_n^{2m+3}\,|\,n,m\in\mathbb{N}\}$ accumulates $\sigma_{2\mathrm{mon}}$-like on $\underline{1}$. Also, for each $n\in\mathbb{N}$: $\{p_n^{m+2}\,|\,m\in\mathbb{N}\}$ accumulates $\sigma_{2\mathrm{mon}}$-like on p_n. To see that $(\sigma,\#)$ is Hausdorff it suffices to check that there are open $U\ni\underline{0}$, $V\ni\underline{1}$ with $U\cap V=\emptyset$, since for other points the verification is easy. We can simply take $U=\{\alpha\in\sigma\,|\,\exists\beta\in\sigma\cap\lessdot 0\gtrdot\;[\alpha\equiv\beta]\}$ and $V=\{\alpha\in\sigma\,|\,\exists\beta\in\sigma\cap\lessdot 1\gtrdot\;[\alpha\equiv\beta]\}$.

So we are reduced to verifying that $(\sigma,\#)$ is NOT T_3. Let U be as above, then U is an open neighborhood of $\underline{0}$. Let V' be an arbitrary open neighborhood of $\underline{0}$, and let W

be open in $(\sigma, \#)$ such that $U \cup W = \sigma$. This means that there is a sequence $(M_n)_{n \in \mathbb{N}}$ in $\mathbb{N}$ such that W contains, popularly speaking, $\{p_n^{M_n+m} \mid n, m \in \mathbb{N}\}$. On the other hand there must be an $N \in \mathbb{N}$ such that V' contains $\{p_{N+n}^{m+2} \mid n, m \in \mathbb{N}\}$. Therefore $W \cap V'$ is inhabited.

REMARK: once again there is a corresponding first enumerable space $(X, \mathcal{T})$ with the same property. For we can take X, D, E the same as in the previous remark 2.0.3 and let $\mathcal{T}$ be the topology generated by the countable basis $D \cup E \cup F' \cup G'$ where

(1) $F' = \{\, \{0\} \cup \{p_{n+N}^{2m+2} \mid n, m \in \mathbb{N}\} \mid N \in \mathbb{N}\,\}$

(iv) $G' = \{\, \{1\} \cup \{p_{n+N}^{2m+3} \mid n, m \in \mathbb{N}\} \mid N \in \mathbb{N}\,\}$

The same remarks concerning $(\sigma, \#)$ and $(X, \mathcal{T})$ hold (mutatis mutandis) as in 2.0.3. The space $(X, \mathcal{T})$ above occurs (mutatis mutandis) already in [Urysohn25a].

2.0.5 QUESTION: is there a $(\sigma, \#)$ which is regular but NOT normal?

2.0.6 EXAMPLE: let σ be a spread, then $(\sigma, \#_\omega)$ is normal. For let U, V be open in $(\sigma, \#_\omega)$, such that $U \cup V = \sigma$. We then have:

($\star$) $\forall \alpha \in \sigma\ \exists s \in \{0,1\}\ [\,(s=0 \wedge \alpha \in U) \vee (s=1 \wedge \alpha \in V)\,]$

By $\mathbf{AC}_{10}$ there is a spread-function γ from σ to $\{0,1\}$ realizing ($\star$). Now put $W = \{\alpha \in \sigma \mid \gamma(\alpha) = 1\}$ and $Z = \{\alpha \in \sigma \mid \gamma(\alpha) = 0\}$. Then W, Z are open in $(\sigma, \#_\omega)$, $W \cap Z = \emptyset$ and $U \cup W = \sigma = Z \cup V$.

2.1 EVERY COMPACT SPACE IS METRIZABLE

2.1.0 in this section we show that every compact space $(X, \mathcal{T})$ is metrizable. We already knew by theorem 1.2.4 that every compact space coincides with an apartness fan. In this section we prove that every apartness fan $(\tau, \#)$ is metrizable. Then we obtain conversely that every apartness fan is compact. The proof that an apartness fan $(\tau, \#)$ is metrizable is by embedding $(\tau, \#)$ in the Hilbert cube $(\mathcal{Q}, d_{\mathcal{Q}})$. This is also the classical strategy used by Urysohn (see [Urysohn25b]) to show that (classically) every first-separable normal space $(X, \mathcal{T})$ is metrizable. Our proof that $(\tau, \#)$ can be embedded in $(\mathcal{Q}, d_{\mathcal{Q}})$ is however along slightly different lines. Observe that we first prove that an apartness fan $(\tau,\#)$ is metrizable, and in this way show $(\tau,\#)$ to be normal and first separable. The ternary real numbers play a nice part in the proof, giving us some unexpected insight in the matters of 'pathwise connected' and 'arcwise connected'. We cannot escape some preparatory notations, definitions and lemmas.

DEFINITION: let $(\sigma,\#)$ be an apartness spread, with corresponding touch-relation $\approx$. Let A, B be subsets of $\overline{\overline{\sigma}}$. We write $A \not\approx B$ for $\forall a\in A\ \forall b\in B\ [a \not\approx b]$, and $A \approx B$ for $\exists a\in A\ \exists b\in B\ [a\approx b]$. For a in $\overline{\overline{\sigma}}$ we then write $a\approx B$, $a\not\approx B$ rather than $\{a\}\approx B$, $\{a\}\not\approx B$.

2.1.1 LEMMA: let $(\sigma,\#)$ be an apartness spread with corresponding touch-relation $\approx$. Suppose for all a, b in $\overline{\overline{\sigma}}$ such that $a\not\approx b$ there is a spread-function $\gamma\in\sigma_\omega$ from $(\sigma,\#)$ to $([0,1],d_{\mathbb{R}})$, such that $\gamma \mid_{\sigma\cap a} \equiv_{\mathbb{R}} 0$ and $\gamma \mid_{\sigma\cap b} \equiv_{\mathbb{R}} 1$. Then $(\sigma,\#)$ is weakly metrizable.

PROOF: there is of course a spread-function f from $(\sigma,\#)$ to $([0,1],d_{\mathbb{R}})$ such that for all α in σ: $f(\alpha)=0$. So if a, b are in $\overline{\overline{\sigma}}$, we can find a spread-function γ from $(\sigma,\#)$ to $([0,1],d_{\mathbb{R}})$, such that

$(\star)$ $\quad a\approx b$ implies $\gamma\equiv_{\mathbb{R}}0$ and $a\not\approx b$ implies $\gamma \mid_{\sigma\cap a} \equiv_{\mathbb{R}} 0$ and $\gamma \mid_{\sigma\cap b} \equiv_{\mathbb{R}} 1$.

This means that we have:

$(\star\star)$ $\quad \forall (a,b)\in\overline{\overline{\sigma}}\times\overline{\overline{\sigma}}\ \exists\gamma\in\sigma_\omega\ [\gamma : (\sigma,\#) \longrightarrow ([0,1],d_{\mathbb{R}}) \wedge (\star)]$

By $\mathbf{AC}_{01}$ there is a sequence $(\gamma_{a,b})_{a,b\in\overline{\overline{\sigma}}}$ realizing $(\star\star)$. Let h be an enumeration of $\overline{\overline{\sigma}}\times\overline{\overline{\sigma}}$. Define a metric d on σ by putting, for α, β in σ: $d(\alpha,\beta)=\sum_{n\in\mathbb{N}} 2^{-n}\cdot|\gamma_{h(n)}(\alpha)-\gamma_{h(n)}(\beta)|$. Then it is easy to see that $d(\alpha,\beta)>0$ iff $\alpha\#\beta$ •

2.1.2 DEFINITION: let $(\sigma,\#)$ be an apartness spread with corresponding touch-relation $\approx$. Call $\approx$ *up-to-date* iff for all a, b in $\overline{\overline{\sigma}}$: $\forall c\in\overline{\overline{\sigma}}\ [(b\sqsubseteq c \wedge lg(c)=lg(a)) \rightarrow a\not\approx c]$ implies $a\not\approx b$.

LEMMA: let $(\tau,\#)$ be an apartness fan. Then there is an up-to-date touch-relation $\approx$ on $\overline{\overline{\tau}}$ such that $\alpha\#\beta$ iff $\exists n\in\mathbb{N}\ [\overline{\alpha}(n)\not\approx\overline{\beta}(n)]$.

PROOF: by proposition 2.0.2 there is a touch-relation $\approx'$ which induces $\#$. Let a, b be in $\overline{\overline{\tau}}$, then $\overline{\overline{\tau}}(lg(a))$ is finite. Therefore we can decide whether $\forall c\in\overline{\overline{\tau}}\ [(b\sqsubseteq c\wedge lg(c)=lg(a)) \rightarrow a\not\approx' c]$ or NOT. In the first case we define $a\not\approx b$, in the second case we define $a\approx b$ •

2.1.3 remember that if σ is a spread and A is a subset of $\overline{\overline{\sigma}}$, then we write σ_A for $\bigcup_{a\in A}\sigma\cap a$, which is a subspread of σ if A is finite and inhabited.

LEMMA: let $(\tau,\#)$ be an apartness fan with a corresponding touch-relation $\approx$ which is up-to-date (definition 2.1.2). Let A, B be finite subsets of $\overline{\overline{\tau}}$ such that for all α in τ_A and all β in τ_B: $\alpha\#\beta$. Then there is an $N\in\mathbb{N}$ such that for all a, b in $\overline{\overline{\tau}}(N)$: $(a\approx A\wedge b\approx B)$ implies $a\not\approx b$.

PROOF: if $A=\emptyset$ or $B=\emptyset$ take $N=0$. Else we have that A, B are finite and inhabited, therefore τ_A and τ_B are subfans of τ.

$\boxed{\text{claim}}$ $\exists M\in\mathbb{N}\ \forall c\in\overline{\overline{\tau}}(M)\ [\,c\not\approx A\vee c\not\approx B\,]$

$\boxed{\text{proof}}$ let γ in τ. For all α in τ_A and all β in τ_B $\alpha\#\beta$, so we have:

$(\star)$ $\forall(\alpha,\beta)\in\tau_A\times\tau_B\ \exists m\in\mathbb{N}\ [\,\overline{\gamma}(m)\not\approx\overline{\alpha}(m)\vee\overline{\gamma}(m)\not\approx\overline{\beta}(m)\,]$

By the fan theorem **FT** we find an $M_0\in\mathbb{N}$ such that for all α in τ_A and all β in τ_B $\overline{\gamma}(M_0)\not\approx\overline{\alpha}(M_0)$ or $\overline{\gamma}(M_0)\not\approx\overline{\beta}(M_0)$. Then since $\overline{\overline{\tau}}_A(M_0)$ and $\overline{\overline{\tau}}_B(M_0)$ are finite, and $\approx$ is up-to-date, we can decide: $\overline{\gamma}(M_0)\not\approx A$ or $\overline{\gamma}(M_0)\not\approx B$. Since γ is arbitrary, we find:

$(\star\star)$ $\forall\gamma\in\tau\ \exists m\in\mathbb{N}\ [\,\overline{\gamma}(m)\not\approx A\vee\overline{\gamma}(m)\not\approx B\,]$

By the fan theorem **FT** we find an $M\in\mathbb{N}$ such that for all γ in τ: $\overline{\gamma}(M)\not\approx A$ or $\overline{\gamma}(M)\not\approx B$ $\circ$

Now let M be as above. Put $A'=A\cup\{a\in\overline{\overline{\tau}}(M)\,|\,a\approx A\}$. $A'\not\approx B$ so by the claim (applied to A' and B) there is $N\in\mathbb{N}$, $N\geq M$ such that for all c in $\overline{\overline{\tau}}(N)$: $c\not\approx A'\vee c\not\approx B$. Then for all a, b in $\overline{\overline{\tau}}(N)$: $a\approx A\wedge b\approx B$ implies $a\not\approx b$ $\bullet$

COROLLARY: in particular, the lemma holds when $A\not\approx B$. Also notice that if $\approx$ is not up-to-date, we still can conclude: there is an $N\in\mathbb{N}$ such that for all a, b in $\overline{\overline{\tau}}(N)$: $(a\approx\overline{\overline{\tau}}_A(N)\wedge b\approx\overline{\overline{\tau}}_B(N))$ implies $a\not\approx b$.

2.1.4* recall the lexicographical ordering $<_{\text{lex}}$ on $\overline{\overline{\sigma}}_\omega$, where $a<_{\text{lex}}b$ iff $a\sqsubset b$ or $\exists i<\text{lg}(a), lg(b)\ [\,\langle\!\langle a_0,\dots a_{i-1}\rangle\!\rangle=\langle\!\langle b_0,\dots b_{i-1}\rangle\!\rangle\wedge a_i<b_i\,]$. For each $n\in\mathbb{N}$, $<_{\text{lex}}$ induces a finite linear ordering on $\overline{\overline{\sigma}}_3(n)$. Now for a in $\overline{\overline{\sigma}}_3(n)$, $a\neq\underline{0}(n)$ we write $Pred(a)$ for the immediate predecessor of a in this finite linear ordering. Similarly, for a in

$\overline{\overline{\sigma}}_3(n)$, $a \neq \underline{2}(n)$ we write $Succ(a)$ for the immediate successor of a. Finally we define: $Pred(\underline{0}(n)) = -1$ and $Succ(\underline{2}(n)) = \lessdot 3 \gtrdot$.

Also recall our definition of $[0,1]_3$, the ternary real numbers in $[0,1]$. In fact $[0,1]_3$ is nothing but $(\sigma_3, \approx_3)$, where for a, b in $\overline{\overline{\sigma}}_3(n)$: $a \approx_3 b$ iff a is in $\{Pred(b), b, Succ(b)\}$.

2.1.5 THEOREM: every apartness fan is metrizable.

PROOF: let $(\tau, \#)$ be an apartness fan. Let $\approx$ be a touch-relation on $\overline{\overline{\tau}}$ corresponding to $\#$. Then let c, e be in $\overline{\overline{\tau}}$, with $c \not\approx e$.

$\boxed{\text{claim}}$ there is a spread-function γ from $(\tau, \#)$ to $([0,1]_3, d_{\mathbb{R}})$ such that $\gamma \restriction_{\tau \cap c} \equiv_{\mathbb{R}} 0$ and $\gamma \restriction_{\tau \cap e} \equiv_{\mathbb{R}} 1$.

$\boxed{\text{proof}}$ put $A_{-1} = \{c\}$ and $A_{\lessdot 3 \gtrdot} = \{e\}$. With induction we define for each $n \in \mathbb{N}$, an $M_n \in \mathbb{N}$ and a partition $\{A_a \,|\, a \in \overline{\overline{\sigma}}_3(n)\}$ of $\overline{\overline{\tau}}(M_n)$ such that:

(i) $\forall a, b \in \overline{\overline{\sigma}}_3(n) \cup \{-1, \lessdot 3 \gtrdot\}\ [A_a \approx A_b \rightarrow a \in \{Pred(b), b, Succ(b)\}]$.

(ii) $A_{b \star \lessdot 0 \gtrdot} = \{c \in \overline{\overline{\tau}}_{A_b}(M_n) \mid c \approx A_{Pred(b)}\}$, for b in $\overline{\overline{\sigma}}_3(n-1)$.

(iii) $A_{b \star \lessdot 1 \gtrdot} = \{c \in \overline{\overline{\tau}}_{A_b}(M_n) \mid A_{Pred(b)} \not\approx c \not\approx A_{Succ(b)}\}$, for b in $\overline{\overline{\sigma}}_3(n-1)$.

(iv) $A_{b \star \lessdot 2 \gtrdot} = \{c \in \overline{\overline{\tau}}_{A_b}(M_n) \mid c \approx A_{Succ(b)}\}$, for b in $\overline{\overline{\sigma}}_3(n-1)$.

(v) M_n is the smallest natural number allowing such partition $\{A_a \,|\, a \in \overline{\overline{\sigma}}_3(n)\}$ of $\overline{\overline{\tau}}(M_n)$.

Basis: $n = 0$. Put $M_0 = 0$ and $A_{\lessdot\gtrdot} = \{\lessdot\gtrdot\}$. Then (i)-(v) are satisfied.
Induction: let $n \in \mathbb{N}$ and suppose $M_n \in \mathbb{N}$ and a partition $\{A_a \,|\, a \in \overline{\overline{\sigma}}_3(n)\}$ of $\overline{\overline{\tau}}(M_n)$ have been defined satisfying (i)-(v). Let a be in $\overline{\overline{\sigma}}_3$(n). If $A_a = \emptyset$ then put $M_a = M_n$. Else consider $\tau' = \tau_{A_{Pred(a)}} \cup \tau_{A_a} \cup \tau_{A_{Succ(a)}}$. $A_{Pred(a)}$, $A_{Succ(a)}$ are subsets of $\overline{\overline{\tau}}'$ such that $A_{Pred(a)} \not\approx A_{Succ(a)}$. Using lemma 2.1.3, determine

$$M_a = \mu m \in \mathbb{N}\, [\forall b, c \in \overline{\overline{\tau}}'(m)\, [b \approx A_{Pred(a)} \wedge c \approx A_{Succ(a)} \rightarrow b \not\approx c]]$$

Put $M_{n+1} = \max(\{M_a \,|\, a \in \overline{\overline{\sigma}}_3(n)\})$. For a in $\overline{\overline{\sigma}}_3(n)$ define unique subsets $A_{a \star \lessdot 0 \gtrdot}$, $A_{a \star \lessdot 1 \gtrdot}$, $A_{a \star \lessdot 2 \gtrdot}$ such that (ii)-(iv) are satisfied (replacing n by $n+1$). Then we find (i)-(v) for M_{n+1} and $\{A_a \,|\, a \in \overline{\overline{\sigma}}_3(n+1)\}$.

We now define γ by specifying, for α in τ: $\gamma(\alpha)$ is the unique β in σ_3 such that for all $n \in \mathbb{N}$: $\overline{\alpha}(M_n) \in A_{\overline{\beta}(n)}$. It is easy to see that γ is a spread-function from $(\tau, \#)$ to $([0,1]_3, d_{\mathbb{R}})$ $\circ$

Therefore $(\tau, \#)$ satisfies the requirements of the Urysohn lemma 2.1.1, and so is weakly metrizable. Let d be a metric which weakly metrizes $(\tau, \#)$. Then (τ, d) is compact, so by theorem 1.2.4 (τ, d) coincides with $(\tau, \#)$, meaning that d metrizes $(\tau, \#)$ •

COROLLARY:

(i) a topological space $(X, \mathcal{T})$ is compact iff $(X, \mathcal{T})$ coincides with an apartness fan.

(ii) every compact space is metrizable.

PROOF: by theorem 1.2.4 every compact space coincides with an apartness fan. The theorem above implies that every apartness fan is Hausdorff, and therefore compact (definition 1.2.2) •

REMARK:

(i) the corollary shows that there are two alternatives to our definition of 'compact', both of them equivalent to this definition. We could define a topological space $(X, \mathcal{T})$ to be compact iff $(X, \mathcal{T})$ coincides with an apartness fan $(\tau, \#)$. Equivalently we could define $(X, \mathcal{T})$ to be compact iff $(X, \mathcal{T})$ coincides with a metric fan. But we feel that the definition given is closest to the classical definition, and is the least ad hoc in character.

(ii) there are connections with [Freudenthal37, thm. 6.14]. For in [Freudenthal37] the metrizability of so-called DFTK-spaces is proved, and also that the class of these DFTK-spaces corresponds precisely to the class of all complete metric fans, which is the class of all strongly compact spaces by our remarks in 1.2.3.

2.1.6 we can extract another interesting phenomenon from the proof of theorem 2.1.5, namely that $([0,1]_3, d_{\mathbb{R}})$ is a pathwise and 1-locally pathwise connected space which is NOT arcwise connected. Remember that we write 0 for the element $\underline{0}$ of $\mathbb{R}$, and 1 for the element $\lessdot 2 \gtrdot \star \underline{0}$ of $\mathbb{R}$.

LEMMA: let f be a continuous function from $([0,1], d_{\mathbb{R}})$ to $([0,1]_3, d_{\mathbb{R}})$ such that $f(0) < \frac{1}{3}$ and $f(1) > \frac{1}{3}$. Then there is a q in $\mathbb{Q} \cap [0,1]$ and an $n \in \mathbb{N}$ such that for all α in $B(q, 2^{-n}) \cap [0,1]$: $f(\alpha) \equiv \frac{1}{3}$.

PROOF: we have:

$(\star)$ $\quad \forall \alpha \in [0,1] \; \exists s \in \{0,1\} \; [\, (s = 0 \wedge f(\alpha) \leq \frac{1}{3}) \vee (s = 1 \wedge f(\alpha) \geq \frac{1}{3}) \,]$

By the fan theorem **FT** there is an $N \in \mathbb{N}$ and a function h from $\overline{\overline{[0,1]}}(N)$ to $\{0,1\}$ such that for all α in $[0,1]$: $h(\overline{\alpha}(N))=0$ implies $f(\alpha) \leq \frac{1}{3}$ and $h(\overline{\alpha}(N))=1$ implies $f(\alpha) \geq \frac{1}{3}$. Then $h(\overline{0}(N))=0$ and $h(\overline{1}(N))=1$, so there are a, b in $\overline{\overline{[0,1]}}_3(N)$, with $a_{\mathbb{R}} < b_{\mathbb{R}}$, such that $h(a)=0$ and $h(b)=1$ and $a \approx_{\mathbb{R}} b$. In fact this means that $b_{\mathbb{R}} - a_{\mathbb{R}} = 3^{-N+1}$. Put $q = \frac{1}{4}(a_{\mathbb{R}} + 3 \cdot b_{\mathbb{R}})$. Then for all α in $B(q, 2^{-2N})$ there are β and γ in $[0,1]$ such that $\overline{\beta}(N)=a$ and $\overline{\gamma}(N)=b$ and $\beta \equiv \alpha \equiv \gamma$. Clearly then for all α in $B(q, 2^{-2N})$ we have: $f(\alpha) \equiv \frac{1}{3}$ •

COROLLARY: $([0,1]_3, d_{\mathbb{R}})$ is NOT arcwise connected.

To see that $([0,1]_3, d_{\mathbb{R}})$ is pathwise connected we use the proof of theorem 2.1.5. Notice that $\overline{0}(2) \not\approx_{\mathbb{R}} \overline{1}(2)$. The proof of theorem 2.1.5 shows how to canonically construct a spread-function γ from $([0,1], d_{\mathbb{R}})$ to $([0,1]_3, d_{\mathbb{R}})$ such that $\gamma(0) \equiv 0$ and $\gamma(1) \equiv 1$.

LEMMA: let α, β be in $[0,1]_3$ such that $\alpha \leq \beta$. Then there is a continuous function f from $([0,1]_3, d_{\mathbb{R}})$ to $([\alpha,\beta]_3, d_{\mathbb{R}})$ such that $f(0)=\alpha$ and $f(1)=\beta$.

PROOF: let δ be in $[0,1]_3$. Define for $n \in \mathbb{N}$:

$$\overline{f(\delta)}(n) \underset{D}{=} \begin{cases} \overline{\alpha}(n) & \text{if } \overline{\delta}(n) <_{\text{lex}} \overline{\alpha}(n) \\ \overline{\beta}(n) & \text{if } \overline{\beta}(n) <_{\text{lex}} \overline{\delta}(n) \\ \overline{\delta}(n) & \text{else} \end{cases}$$

This completely describes f. It is easy to see that f is as required •

COROLLARY: let α, β be in $[0,1]_3$ such that $\alpha \leq \beta$ or $\alpha \geq \beta$. Then there is a continuous function g from $([0,1], d_{\mathbb{R}})$ to $([0,1]_3, d_{\mathbb{R}})$ such that $g(0) \equiv \alpha$ and $g(1) \equiv \beta$.

PROOF: first suppose $\alpha \leq \beta$. Let γ be the spread-function from $([0,1], d_{\mathbb{R}})$ to $([0,1]_3, d_{\mathbb{R}})$ described above, such that $\gamma(0) \equiv 0$ and $\gamma(1) \equiv 1$. Let f be a continuous function from $([0,1]_3, d_{\mathbb{R}})$ to $([\alpha,\beta]_3, d_{\mathbb{R}})$ such that $f(0)=\alpha$ and $f(1)=\beta$. Simply put $g = f \circ \gamma$, then g satisfies the corollary. Now suppose $\alpha \geq \beta$. Then by the above there is a continuous function g' from $([0,1], d_{\mathbb{R}})$ to $([0,1]_3, d_{\mathbb{R}})$ such that $g'(0) \equiv \beta$ and $g'(1) \equiv \alpha$. Let h be the function from $([0,1], d_{\mathbb{R}})$ to $([0,1], d_{\mathbb{R}})$ given by $h(\kappa) = -\kappa + 1$, for κ in $[0,1]$. Then the function $g = g' \circ h$ satisfies the corollary •

PROPOSITION: $([0,1]_3, d_{\mathbb{R}})$ is pathwise connected.

PROOF: let α, β be in $[0,1]_3$. Without loss of generality $\alpha(0)=\beta(0)$. We must find a continuous function f from $([0,1], d_{\mathbb{R}})$ to $([0,1]_3, d_{\mathbb{R}})$ such that $f(0) \equiv \alpha$ and $f(1) \equiv \beta$. Suppose $n \in \mathbb{N}$ is such that $\overline{\alpha}(n) = \overline{\beta}(n)$. Then we put $\overline{f(\kappa)}(n) = \overline{\alpha}(n)$ for all κ in $[0,1]$. As soon as we encounter an $m \in \mathbb{N}$ such that $\overline{\alpha}(m) = \overline{\beta}(m)$ and $\overline{\alpha}(m+1) \neq \overline{\beta}(m+1)$, then we can apply the corollary above to find a continuous function g from $([0,1], d_{\mathbb{R}})$ to $([0,1]_3 \cap \overline{\alpha}(m), d_{\mathbb{R}})$ such that $g(0) \equiv \alpha$ and $g(1) \equiv \beta$. This since $[0,1]_3 \cap \overline{\alpha}(m)$ is the same as $\{\overline{\alpha}(m) \star \nu \mid \nu \in \sigma_3\}$, an almost literal copy of $[0,1]_3$ itself. By lemma 0.1.5 without loss of generality g is a spread-function from $[0,1]$ to $[0,1]_3 \cap \overline{\alpha}(m)$. This means that we can continue our f by putting: $\overline{f(\kappa)}(m+n) = \overline{g(\kappa)}(n)$ for $n \in \mathbb{N}$. This means that we can construct f as a continuous spread-function from $([0,1], d_{\mathbb{R}})$ to $([0,1]_3, d_{\mathbb{R}})$. Clearly $f(0) \equiv \alpha$ and $f(1) \equiv \beta$ •

By carefully reading the above the reader may convince her- or himself that $([0,1]_3, d_{\mathbb{R}})$ is also locally pathwise connected, but NOT locally arcwise connected.

2.2 1-LOCALLY COMPACT SPACES

2.2.0 in this section we analyze the concept of a 1-locally compact topological space. Clearly it suffices to analyze the concept of a locally compact topological spread $(\sigma, \mathcal{T})$. But by theorem 1.4.1 a locally compact $(\sigma, \mathcal{T})$ coincides identically with $(\sigma, \#)$. Therefore it suffices to consider locally compact apartness spreads $(\sigma, \#)$.

We show amongst others: every compact space is 1-locally compact; every locally compact $(\sigma, \#)$ has a one-point compact extension and is therefore metrizable; an apartness spread $(\sigma, \#)$ is locally strongly compact iff $(\sigma, \#)$ is locally compact and topologically complete iff $(\sigma, \#)$ admits a one-point compact extension which is topologically complete. Except for the metrizability, these results are obtained using only the apartness topology, and not by using theorem 2.1.5.

2.2.1 LEMMA: let $(\sigma, \#)$ be an apartness fan. Let τ be a subfan of σ. Then $V = \{\alpha \in \sigma \mid \forall \beta \in \tau\, [\alpha \# \beta]\}$ is open in $(\sigma, \#)$.

PROOF: let $\approx$ be an up-to-date touch-relation on σ corresponding to $\#$. Let α be in V. We have:

$(\star)$ $\quad \forall \beta \in \tau\; \exists n \in \mathbb{N}\; [\overline{\alpha}(n) \not\approx \overline{\beta}(n)]$

By the fan theorem **FT** there is an $N\in\mathbb{N}$ such that for all a in $\overline{\overline{\tau}}(N)$: $a \not\approx \overline{\alpha}(N)$. Now apply lemma 2.1.3 (with $A=\overline{\overline{\tau}}(N)$ and $B=\{\overline{\alpha}(N)\}$) to find an $M\in\mathbb{N}$ such that for all b in $\overline{\overline{\sigma}}(M)$: $b \not\approx \overline{\overline{\tau}}(N)$ or $b \not\approx \overline{\alpha}(N)$. Clearly we can now decide, for all γ in σ: $\gamma \# \alpha$ or $\gamma \in V$. Since α is arbitrary, this shows that V is open in $(\sigma, \#)$ •

2.2.2 PROPOSITION: every compact space is 1-locally compact.

PROOF: by theorem 1.2.4 it suffices to prove that every apartness fan is 1-locally compact. Let $(\tau, \#)$ be an apartness fan. Let α be in τ, and let U be an open neighborhood of α in $(\tau, \#)$. We must come up with a compact neighborhood W of α such that $W \subseteq U$. But we have:

(⋆) $\forall \beta\in\tau\ \exists s\in\{0,1\}\ [\,(s=0 \wedge \beta\#\alpha)\vee(s=1 \wedge \beta\in U)\,]$

By the fan theorem **FT** there is an $N\in\mathbb{N}$ and a function h from $\overline{\overline{\tau}}(N)$ to $\{0,1\}$ such that for all β in τ: $h(\overline{\beta}(N))=0$ implies $\beta\#\alpha$ and $h(\overline{\beta}(N))=1$ implies $\beta\in U$. Put $\tau_0=\{\beta\in\tau \mid h(\overline{\beta}(N))=0\}$ and $\tau_1=\{\beta\in\tau \mid h(\overline{\beta}(N))=1\}$. Put $W=\{\beta\in\tau \mid \exists\gamma\in\tau_1\,[\beta\equiv\gamma]\,\}$. Clearly $W\subseteq U$, and also: $(W, \mathcal{T}_W)$ is fanlike and Hausdorff, therefore compact. It only remains to verify that W is a neighborhood of α in $(\tau, \#)$. Define a subset V of W by putting:

$V=\{\beta\in\tau \mid \forall\gamma\in\tau_0\,[\beta\#\gamma]\,\}$.

Clearly α is in V, and by lemma 2.2.1 V is open in $(\tau, \#)$ Therefore W is a neighborhood of α in $(\tau, \#)$ •

REMARK: keep in mind that our definition of ‘compact’ entails the Hausdorff property. See example 1.4.4 for a weakly compact T_1 space which is NOT locally weakly compact.

2.2.3 LEMMA: let $(X, \#)$ be an apartness space, and let τ be a subfan of X. Put $A=\{y\in X \mid \exists z\in\tau\ [\,y\equiv z\,]\}$. Then $(A, \mathcal{T}_A)$ coincides identically with $(A, \#)$.

PROOF: by 1.0.3 it suffices to show that $(A, \mathcal{T}_A)$ refines $(A, \#)$. To this end, let $\approx$ be a touch-relation on τ corresponding to $\#$ (restricted to τ). Let U be open in $(A, \#)$, and let α be in U. We must come up with an open V in $(X, \#)$ such that $V\cap A\subseteq U$. Since U is open in $(A, \#)$ we find:

(⋆) $\forall\beta\in\tau\ \exists(s,n)\in\{0,1\}\times\mathbb{N}\ [\,(s=0 \wedge \beta\in U)\vee(s=1 \wedge \overline{\beta}(n) \not\approx \overline{\alpha}(n))\,]$

By the fan theorem **FT** there is an $N\in\mathbb{N}$ and a function h from $\overline{\overline{\tau}}(N)$ to $\{0,1\}$ such that for each β in τ: $h(\overline{\beta}(N))=0$ implies $\beta\in U$ and $h(\overline{\beta}(N))=1$ implies $\overline{\beta}(N)\not\approx\overline{\alpha}(N)$.

$\boxed{\text{case 1}}$ $\forall a\in\overline{\overline{\tau}}(N)\ [h(a)=0]$.
Then we can take $V=X$!

$\boxed{\text{case 2}}$ $\exists a\in\overline{\overline{\tau}}(N)\ [h(a)=1]$.
Put $B=\{a\in\overline{\overline{\tau}}(N)\,|\,h(a)=1\}$. Then τ_B is a subfan of τ. Put $V=\{y\in X\,|\,\forall\beta\in\tau_B\ [y\#\beta]\}$. It is easy to see that $V\cap A\subseteq U$. We claim that V is open in $(X,\#)$. To this end let γ be in V, and let δ be arbitrary in X. We must show that we can decide: $\delta\#\gamma$ or $\delta\in V$. Put $\tau'=\tau_B\cup\{\alpha\in\tau\mid\forall n\in\mathbb{N}\ [\overline{\alpha}(n)\in\{\overline{\gamma}(n),\overline{\delta}(n)\}]\}$, then τ' is a fan. By lemma 2.2.1 $V\cap\tau'$ is open in $(\tau',\#)$. But γ is in $V\cap\tau'$, so we can decide: $\delta\#\gamma$ or $\delta\in V$ •

2.2.4 THEOREM: for a topological space $(X,\mathcal{T})$ the following conditions are equivalent:

(i) $(X,\mathcal{T})$ is 1-locally compact.

(ii) $(X,\mathcal{T})$ admits an enumerable cover of compact neighborhoods.

(iii) there is a sequence $(\tau_n)_{n\in\mathbb{N}}$ of fans and a Σ^1_0-apartness $\#$ on $\sigma=\biguplus_{n\in\mathbb{N}}\tau_n$ such that $(X,\mathcal{T})$ coincides with $(\sigma,\#)$ and moreover: $\{\tilde{\tau}_n\,|\,n\in\mathbb{N}\}$ is a cover of $(\sigma,\#)$, where for $n\in\mathbb{N}$: $\tilde{\tau}_n=\{\alpha\in\sigma\,|\,\exists\beta\in\tau_n\,[\alpha\equiv\beta]\}$.

PROOF: remember that a topological space is compact iff it coincides with an apartness fan (corollary 2.1.5). First we show that (ii) implies (i) and (iii). For this we need proposition 2.0.2, which says that a Σ^1_0-apartness on a spread σ is determined by a touch-relation $\approx$ on $\overline{\overline{\sigma}}$. Notice that we can code any such touch-relation with an element $\delta_\approx$ of σ_2. Secondly we use lemma 0.1.5, which says that a function from a topological spread to an arbitrary space can be represented by a spread-function. This shows that a compact space $(Y,\mathcal{T}')$ can be completely represented by an element $(\tau,\delta_\approx,\gamma)$ of $\sigma_\omega\times\sigma_\omega\times\sigma_\omega$, where $(\tau,\#_{\delta_\approx})$ is an apartness fan and γ is a homeomorphism from $(\tau,\#_{\delta_\approx})$ to $(Y,\mathcal{T}')$. This observation will allow us to apply $\mathbf{AC}_{01}$.

Suppose that $(W_n)_{n\in\mathbb{N}}$ is a sequence of subsets of $(X,\mathcal{T})$ such that $\{W_n\,|\,n\in\mathbb{N}\}$ is a cover of $(X,\mathcal{T})$ and for each $n\in\mathbb{N}$: $(W_n,\mathcal{T}_{W_n})$ is compact. By lemma 1.1.1 (ii) we have that $(W_n,\mathcal{T}_{W_n})$ coincides identically with $(W_n,\#)$. By lemma 1.4.1 above, $(X,\mathcal{T})$ then coincides identically with $(X,\#)$. By proposition 2.0.2 and lemma 0.1.5 we see:

$(\star)$ $\forall n\in\mathbb{N}\ \exists\tau,\delta_\approx,\gamma\in\sigma_\omega$ [γ is a homeomorphism from the apartness fan $(\tau,\#_{\delta_\approx})$ to $(W_n,\#)$]

By $\mathbf{AC}_{01}$ we obtain a sequence of apartness fans $((\tau_n, \#_n))_{n\in\mathbb{N}}$ and a sequence of spread-functions $(\gamma_n)_{n\in\mathbb{N}}$ such that for each $n\in\mathbb{N}$: γ_n is a homeomorphism from $(\tau_n, \#_n)$ to $(W_n, \#)$. In fact $(\gamma_n)_{n\in\mathbb{N}}$ gives us a surjection γ from $(\biguplus_{n\in\mathbb{N}}\tau_n, d_\omega)=(\sigma, d_\omega)$ to $(X, \mathcal{T})$. Define $\#$ on σ by putting, for α, β in σ: $\alpha\#\beta$ iff $\gamma(\alpha)\#\gamma(\beta)$. Then $(X, \#)$ coincides with $(\sigma, \#)$, meaning $(X, \#)$ is spreadlike. Next, let x be in X, and let $U\ni x$ be open in $(X, \#)$. Determine $n\in\mathbb{N}$ such that $(W_n, \#)$ is a neighborhood of x. Determine V open in $(X, \#)$ such that $x\in V\subseteq W_n$ then $U\cap V\subseteq W_n$ and $x\in U\cap V$. But $(W_n, \#)$ is locally compact by proposition 2.2.2 above, so we can find a Z open in $(W_n, \#)$ and a compact subspace $(W, \#)$ of $(W_n, \#)$ such that $x\in Z\subseteq W\subseteq (U\cap V)$. There is a Z', open in $(X, \#)$, such that $Z=Z'\cap W_n$. Then $Z'\cap U\cap V$ is open in $(X, \#)$, and $x\in Z'\cap U\cap V$, and $Z'\cap U\cap V\subseteq W$. This shows that $W\subseteq U$ is a compact neighborhood of x in $(X, \#)$. Since x and U are arbitrary, we obtain both L_0 and L_1 for $(X, \#)$ with respect to 'compact', proving (i). Notice that we have also proved (iii) in the process.

Next suppose (iii). We show that this implies (ii). It clearly suffices to show that for $n\in\mathbb{N}$: $(\tilde{\tau}_n, \mathcal{T}_{\tilde{\tau}_n})$ is compact. But by lemma 2.2.3 we see that $(\tilde{\tau}_n, \mathcal{T}_{\tilde{\tau}_n})$ coincides with $(\tau_n, \#)$. Then we are done by corollary 2.1.5 (ii).

Finally suppose (i). We resort to our analysis above, which tells us that a compact space can be coded by an element of σ_ω. Let h be a homeomorphism from $(X, \mathcal{T})$ to a topological spread $(\sigma, \mathcal{T})$. Then $(\sigma, \mathcal{T})$ satisfies L_0 with respect to 'compact'. Let α be in σ. Then since $\{\sigma\}$ is an open cover of $(\sigma, \mathcal{T})$, by L_0 (using proposition 2.0.2 and lemma 0.1.5) there is a $(\tau, \delta_\approx, \gamma)\in\sigma_\omega\times\sigma_\omega\times\sigma_\omega$ such that:

(★★) γ is a homeomorphical embedding of the apartness fan $(\tau, \#_{\delta_\approx})$ in $(\sigma, \mathcal{T})$ with the property that $\gamma(\tau)$ is a neighborhood of α in $(\sigma, \mathcal{T})$.

So there is a U open in $(\sigma, \mathcal{T})$ with $\alpha\in U\subseteq\gamma(\tau)$. Also, (σ, d_ω) refines $(\sigma, \mathcal{T})$ (see 1.0.4). Therefore there is an $n\in\mathbb{N}$ such that $\sigma\cap\overline{\alpha}(n)\subseteq U$. We conclude:

$$\mathcal{U}=\{\sigma\cap\overline{\alpha}(n) \mid \alpha\in\sigma, n\in\mathbb{N} \mid \exists\tau, \delta_\approx, \gamma\in\sigma_\omega\ \forall\beta\in\sigma\cap\overline{\alpha}(n)\ [(\tau, \delta_\approx, \gamma) \text{ realizes } (\star\star) \text{ for } \beta]\}$$

is an open cover of (σ, d_ω). By proposition 1.1.6 (using $\mathbf{AC}_{10}$) there is an enumerated refinement $\mathcal{V}=\{V_n \mid n\in\mathbb{N}\}$ of $\mathcal{U}$ with respect to (σ, d_ω). Combining a little we find:

$$(*)\quad \forall n\in\mathbb{N}\ \exists\tau, \delta_\approx, \gamma\in\sigma_\omega\ [\forall\beta\in V_n\,[(\tau, \delta_\approx, \gamma) \text{ realizes } (\star\star) \text{ for } \beta]]$$

By $\mathbf{AC}_{01}$ we obtain a sequence $(\tau_n, \delta_{\approx n}, \gamma_n)_{n\in\mathbb{N}}$ in $\sigma_\omega\times\sigma_\omega\times\sigma_\omega$ realizing $(*)$. Clearly $\{\gamma_n(\tau_n) \mid n\in\mathbb{N}\}$ is an enumerable cover of $(\sigma, \mathcal{T})$ with compact neighborhoods, therefore $\{h^{-1}(\gamma_n(\tau_n)) \mid n\in\mathbb{N}\}$ is an enumerable cover of $(X, \mathcal{T})$ with compact neighborhoods •

2.2.5 DEFINITION: let $(X,\mathcal{T})$ be a topological space, and let $(Y,\mathcal{T}')$ be a compact space. Then $(Y,\mathcal{T}')$ is a *compact extension* of $(X,\mathcal{T})$ iff $(X,\mathcal{T})$ coincides with a subspace of $(Y,\mathcal{T}')$. $(Y,\mathcal{T}')$ is called a *one-point compact extension* of $(X,\mathcal{T})$ iff in addition there is an y in Y such that $(X,\mathcal{T})$ coincides with $(\{z\in Y \mid z\#y\},\#)$. $(Y,\mathcal{T}')$ is called a *compactification* of $(X,\mathcal{T})$ iff $(X,\mathcal{T})$ coincides with a dense subspace of $(Y,\mathcal{T}')$. $(Y,\mathcal{T}')$ is called a *one-point compactification* of $(X,\mathcal{T})$ iff in addition there is an y in Y such that $(X,\mathcal{T})$ coincides with $(\{z\in Y \mid z\#y\},\#)$.

We wish to prove that each 1-locally compact space has a one-point compact extension. For this we need to refine theorem 2.2.4, showing that for a 1-locally compact space $(X,\mathcal{T})$, X can be written as a union $\bigcup_{n\in\mathbb{N}}W_n$ of subsets of X such that firstly: for each $n\in\mathbb{N}$ $W_n=\emptyset$ or $(W_n,\mathcal{T}_{W_n})$ is compact, and secondly for all $n,m\in\mathbb{N}$, $n\notin\{m-1,m,m+1\}$: for all α,β in W_n, W_m respectively we have that $\alpha\#\beta$.

THEOREM: let $(X,\mathcal{T})$ be 1-locally compact. Then $(X,\mathcal{T})$ coincides with an apartness spread $(\rho,\#)$ with corresponding touch-relation $\approx$ such that :

(i) $\forall n\in\mathbb{N}\,[\,\rho(\lessdot n\gtrdot)=0 \rightarrow \rho\cap\lessdot n\gtrdot$ is a fan$\,]$.

(ii) $\forall n,m\in\mathbb{N}\,[\,(\rho(\lessdot n\gtrdot)=0=\rho(\lessdot m\gtrdot)\wedge|n-m|>1) \rightarrow \lessdot n\gtrdot \not\approx \lessdot m\gtrdot\,]$.

PROOF: we copy the notations from theorem 2.2.4 above, so then $(X,\mathcal{T})$ coincides with $(\sigma,\#)$ etc. For $n\in\mathbb{N}$ let ρ_n be the fan $\biguplus_{i\leq n}\tau_i$ (we identify ρ_n with the subset $\cup_{i\leq n}\tau_i$ of σ), and put $\tilde\rho_n=\{\alpha\in\sigma\,|\,\exists\beta\in\rho_n\,[\alpha\equiv\beta]\,\}$. Clearly $(\tilde\rho_n)_{n\in\mathbb{N}}$ is a cover of $(\sigma,\#)$. Let $n\in\mathbb{N}$. We have:

$(\star)$ $\forall\alpha\in\rho_n\ \exists m\in\mathbb{N}\,[\,\tilde\rho_m$ is a neighborhood of α in $(\sigma,\#)\,]$

By the fan theorem **FT** we see that there is $M\in\mathbb{N}$ such that for all α in ρ_n: $\tilde\rho_M$ is a neighborhood of α in $(\sigma,\#)$. So we find:

$(\star\star)$ $\forall n\in\mathbb{N}\ \exists M\in\mathbb{N}\ \forall\alpha\in\rho_n\,[\,\tilde\rho_M$ is a neighborhood of $\alpha\,]$

By $\mathbf{AC}_{01}$ there is a function h from $\mathbb{N}$ to $\mathbb{N}$ realizing $(\star\star)$. Define a sequence $(M_n)_{n\in\mathbb{N}}$ inductively, putting $M_0=0$ and for $n\in\mathbb{N}$: $M_{n+1}=h(M_n)+n$. Now let $t\in\mathbb{N}_{\geq 2}$.

$\boxed{\text{claim}}$ $\forall\alpha\in\rho_{M_t}\ \exists p\in\{0,1\}\,[\,(p=0\wedge\alpha\in\tilde\rho_{M_{t-1}})\vee(p=1\wedge\forall\beta\in\tilde\rho_{M_{t-2}}\,[\alpha\#\beta])\,]$.

$\boxed{\text{proof}}$ let α be in ρ_{M_t}. Since for all β in $\rho_{M_{t-2}}$: $\tilde\rho_{M_{t-1}}$ is a neighborhood of β in $(\sigma,\#)$, we obtain:

$$(*) \quad \forall \beta \in \rho_{M_{t-2}}\ \exists s \in \{0,1\}\ [\,(s=0 \wedge \alpha \in \tilde{\rho}_{M_{t-1}}) \vee (s=1 \wedge \alpha \# \beta)\,]$$

By the fan theorem **FT** there is an $N \in \mathbb{N}$ and a function k from $\overline{\overline{\rho}}_{M_{t-2}}(N)$ to $\{0,1\}$ such that for β in $\rho_{M_{t-2}}$: $k(\overline{\beta}(N))$ realizes $(*)$. So if we take p equal to $\min(\{k(a) \mid a \in \overline{\overline{\rho}}_{M_{t-2}}(N)\}$, then p realizes the claim for α $\circ$

Applying the fan theorem **FT** to the claim, we see that there is a $K \in \mathbb{N}$ and a function l from $\overline{\overline{\rho}}_{M_t}(K)$ to $\{0,1\}$ such that for all α in ρ_{M_t}: $l(\overline{\alpha}(K))$ realizes the claim for α. Notice that l is finite, and can be coded by a natural number. Since t is arbitrary we find:

$$(**) \quad \forall t \in \mathbb{N}_{\geq 2}\ \exists K \in \mathbb{N}\ \exists l \in \{0,1\}^{\overline{\overline{\rho}}_{M_t}(K)}\ \forall \alpha \in \rho_{M_t}\ [\,l(\overline{\alpha}(K)) \text{ realizes the claim for } \alpha\,]$$

Using $\mathbf{AC}_{00}$ we obtain a sequence $(K_t, l_t)_{t \in \mathbb{N}_{\geq 2}}$ in $\mathbb{N} \times \mathbb{N}$ realizing $(**)$. For $t \in \mathbb{N}_{\geq 2}$ and $a \in \overline{\overline{\rho}}_{M_t}(K_t)$ we could call the subfan $\rho_{M_t} \cap a$ *old* if $l_t(a)=0$ and *new* if $l_t(a)=1$. The promised spread ρ now arises from collecting ρ_{M_1} and all new subfans of $(\rho_{M_t})_{t \in \mathbb{N}_{\geq 2}}$. To be precise: let a be in $\mathbb{N}$, then:

$$\rho(a) \underset{D}{=} \begin{cases} 0 & \text{if } a = \langle\!\langle\,\rangle\!\rangle \\ 0 & \text{if } a_0 = \langle\!\langle 1 \rangle\!\rangle \text{ and } \rho_{M_1}(\langle\!\langle a_1 \ldots, a_{lg(a)-1} \rangle\!\rangle) = 0 \\ 0 & \text{if } a = \langle\!\langle t \rangle\!\rangle \star b \text{ for } t \in \mathbb{N}_{\geq 2},\, b \in \overline{\overline{\rho}}_{M_t} \text{ such that } \exists c \in \overline{\overline{\rho}}_{M_t}(K_t)\ [\,b \sqsubseteq c \wedge l_t(c)=1\,] \\ 1 & \text{else} \end{cases}$$

We define a surjection j from $(\rho, \#_\omega)$ to $(\sigma, \#)$ by putting: $j(\langle\!\langle n \rangle\!\rangle \star \alpha) = \alpha$, for $\langle\!\langle n \rangle\!\rangle \star \alpha$ in ρ. We define an apartness $\#$ on ρ by putting: $\alpha \# \beta$ iff $j(\alpha) \# j(\beta)$, for α, β in ρ. By proposition 2.0.2 there is a touch-relation $\approx$ on $\overline{\overline{\rho}}$ corresponding to the apartness $\#$ on ρ. It is easy to see that we can take $\approx$ such that the theorem is satisfied $\bullet$

COROLLARY: every 1-locally compact space has a one-point compact extension, and is metrizable.

PROOF: copying notations from above, we define a fan τ as follows. Let a be in $\mathbb{N}$, then:

$$\tau(a) \underset{D}{=} \begin{cases} 0 & \text{if } a = \underline{\overline{0}}(lg(a)) \\ 0 & \text{if } a = \underline{\overline{0}}(c_0) \star c \text{ for certain } c \text{ in } \overline{\overline{\rho}} \\ 1 & \text{else} \end{cases}$$

Put $A = \{\alpha \in \tau \mid \alpha \#_\omega \underline{0}\}$. Define a surjection i from $(A, \#_\omega)$ to $(\rho, \#)$ by putting $i(\underline{\overline{0}}(n) \star \langle\!\langle n \rangle\!\rangle \star \alpha) = \langle\!\langle n \rangle\!\rangle \star \alpha$, for $\underline{\overline{0}}(n) \star \langle\!\langle n \rangle\!\rangle \star \alpha$ in A. We use i to define an apartness $\#$ on τ as follows. For α, β in A put $\alpha \# \beta$ iff $i(\alpha) \# i(\beta)$, and also put $\alpha \# \underline{0}$. This completely determines $\#$ on τ, as is readily verified. Clearly $(\rho, \#)$ coincides with $(A, \#)$ showing that $(\tau, \#)$ is a one-point compact extension of $(\rho, \#)$ and so of $(X, \mathcal{T})$. Trivially A is open in $(\tau, \#)$. By lemma 1.3.1 $(A, \#)$ coincides with $(A, \mathcal{T}_A)$, the subspace of $(\tau, \#)$ with the subspace topology. By theorem 2.1.5 there is a metric d on τ such that $(\tau, \#)$ coincides with (τ, d). Then the subspace $(A, \mathcal{T}_A)$ coincides with (A, d). This shows that $(X, \mathcal{T})$ is metrizable •

2.2.6 theorem 2.2.5 enables us to define a convenient metric d_{lsup} on the set $C((X, \mathcal{T}), (Y, d_Y))$ of all continuous spread-functions from a 1-locally compact space $(X, \mathcal{T})$ to a metric space (Y, d_Y). Be careful, since the resulting $(C((X, \mathcal{T}), (Y, d_Y)), d_{\text{lsup}})$ is not always a *separable* metric space (see proposition 0.5.5).

DEFINITION: let $(\rho, \#)$ be an apartness spread with corresponding touch-relation $\approx$ such that :

(i) $\forall n \in \mathbb{N}\, [\, \rho(\lessdot n \gtrdot) = 0 \rightarrow \rho \cap \lessdot n \gtrdot$ is a fan$\,]$.

(ii) $\forall n, m \in \mathbb{N}\, [\, (\rho(\lessdot n \gtrdot) = 0 = \rho(\lessdot m \gtrdot) \wedge |n - m| > 1) \rightarrow \lessdot n \gtrdot \not\approx \lessdot m \gtrdot\,]$.

Let f and g be two spread-functions from $(\rho, \#)$ to (Y, d_Y), a metric space. For $n \in \mathbb{N}$ write f_n, g_n for the restriction of f, g to $\rho \cap \lessdot n \gtrdot$. Put $d_{\text{sup}}(f_n, g_n) = 0$ if $\rho \cap \lessdot n \gtrdot = \emptyset$. Then $d_{\text{lsup}}(f, g) \underset{D}{=} \sum_{n \in \mathbb{N}} 2^{-n} \cdot \frac{d_{\text{sup}}(f_n, g_n)}{1 + d_{\text{sup}}(f_n, g_n)}$.

By theorem 2.2.5 this shows, for a 1-locally compact space $(X, \mathcal{T})$, how to define d_{lsup} on $C((X, \mathcal{T}), (Y, d_Y))$. We have: $(C((X, \mathcal{T}), (Y, d_Y)), d_{\text{lsup}})$ refines $(C((X, \mathcal{T}), (Y, d_Y)), d_{\text{dense}})$ (see 0.5.7). The advantage of d_{lsup} over d_{dense} is that if (Y, d_Y) is a complete metric space, then $(C((X, \mathcal{T}), (Y, d_Y)), d_{\text{lsup}})$ is a complete metric space. By a similar argument as in 0.5.6, if (Y, d_Y) is a locally convex linear space, then $(C((X, \mathcal{T}), (Y, d_Y)), d_{\text{lsup}})$ is a linear space.

2.2.7 we continue our somewhat exhaustive analysis with an alternative characterization of 1-locally strongly compact.

PROPOSITION: a topological space $(X, \mathcal{T})$ is 1-locally strongly compact iff it is 1-locally compact and topologically complete.

PROOF: since either way $(X, \mathcal{T})$ is 1-locally compact, we copy the notations from the proofs of theorem and corollary 2.2.5. Then $(X, \mathcal{T})$ coincides with the apartness spread

$(\sigma,\#)=(\dot{\bigcup}_{n\in\mathbb{N}}\tau_n,\,\#)$ etc. For the implication from right to left, let $(\sigma,\#)$ be (locally compact and) topologically complete. Let d be a metric on σ such that (σ,d) is complete and coincides identically with $(\sigma,\#)$. To show that $(\sigma,\#)$ is locally strongly compact, let β in σ. It suffices to show that there is a strongly compact neighborhood of β in $(\sigma,\#)$. Determine $n,m\in\mathbb{N}$ with $B(\beta,2^{-n})\subseteq\tilde{\tau}_m$. By theorem 0.4.7 there is a $\delta\in\mathbb{R}^+$, $\delta<2^{-n}$ such that $(cB(\beta,\delta),d)$ is compact. Trivially $(cB(\beta,\delta),d)$ is complete, and so a strongly compact neighborhood of β in $(\sigma,\#)$.

For the other implication, let $(\sigma,\#)$ be locally strongly compact. Then without loss of generality for $n\in\mathbb{N}$ $(\tau_n,\#)$ is strongly compact. We copy the notations from the proof of corollary 2.2.5. Define a d-equivalent metric d' on A by putting $d'(\alpha,\beta)=d(\alpha,\beta)+d_{\mathbb{R}}(\frac{1}{d(\alpha,\underline{0})},\frac{1}{d(\beta,\underline{0})})$, for α and β in A. Define d' on σ by putting $d'(\alpha,\beta)=d'(j^{-1}\circ i^{-1}(\alpha),j^{-1}\circ i^{-1}(\beta))$, for α and β in σ (j is the homeomorphism from $(\rho,\#)$ to $(\sigma,\#)$, and i is the homeomorphism from (A,d) to $(\rho,\#)$).

$\boxed{\text{claim}}$ (σ,d') is complete.

$\boxed{\text{proof}}$ let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy-sequence in (σ,d'). It suffices to show that $(a_n)_{n\in\mathbb{N}}$ converges in (σ,d'). It is straightforward to check that there is $N\in\mathbb{N}$ such that for all $n\in\mathbb{N}$: a_n is in $\tilde{\rho}_N$. We have:

$(\star)$ $\forall n\in\mathbb{N}\;\exists i\leq N\;[\,a_n\in\tilde{\tau}_i\,]$

Use $\mathbf{AC}_{00}$ to determine a function h from $\mathbb{N}$ to $\{i\in\mathbb{N}\,|\,i\leq N\}$ realizing $(\star)$. For $i\leq N$ determine b_i in τ_i, and inductively define a Cauchy-sequence $(c_{i,s})_{s\in\mathbb{N}}$ in (τ_i,d') by putting $c_{i,0}=b_i$ and for $s\in\mathbb{N}$:

$$c_{i,s+1}\underset{D}{=}\begin{cases} c_{i,s} & \text{if } h(s+1)\neq i\\ a_{s+1} & \text{else}\end{cases}$$

But since $(\tau_i,\#)$ is strongly compact, (τ_i,d') is complete, for $i\leq N$ (see 1.2.3). So for $i\leq N$ let $\alpha_i\in\tau_i$ be the d'-limit of $(c_{i,s})_{s\in\mathbb{N}}$. Since $\{\tilde{\tau}_n\,|\,n\in\mathbb{N}\}$ is a cover of $(\sigma,\#)$, we can determine for each $i\leq N$: $n_i\in\mathbb{N}$ and $\delta_i\in\mathbb{R}^+$ such that $B(\alpha_i,\delta_i)\subseteq\tilde{\tau}_{n_i}$. But now we obviously can find a specific $i\leq N$ and an $M\in\mathbb{N}$ such that for all $n\in\mathbb{N},n>M$: a_n is in $B(\alpha_i,\delta_i)\subseteq\tilde{\tau}_{n_i}$. By 1.2.3 $(\tilde{\tau}_{n_i},d')$ is complete, meaning $(a_n)_{n\in\mathbb{N}}$ converges in (σ,d') ∘ •

COROLLARY: a topological space $(X,\mathcal{T})$ is 1-locally strongly compact iff $(X,\mathcal{T})$ admits a one-point compact extension which is strongly compact.

PROOF: we leave the proof to the reader •

REMARK: in fact the following holds for (σ, d'). Let B be an inhabited subset of (σ, d') such that there is an $N \in \mathbb{N}$ with: $\forall a, b \in B\, [\, d'(a,b) < N\,]$. Then there is a strongly compact subspace (W, d') of (σ, d') such that $B \subseteq W$. This metric phenomenon is essentially equivalent to the notion of 'locally compact' in Bishop's school. It merits a definition.

DEFINITION: let (X, d) be a metric space. Let A be a subset of (X, d). We say that A is *bounded* iff there is an $N \in \mathbb{N}$ with: $\forall a, b \in A\, [\, d(a,b) < N\,]$. We say that (X, d) is *boundedly strongly compact* iff each inhabited bounded subset of (X, d) is contained in a strongly compact subspace of (X, d).

COROLLARY: a topological space $(X, \mathcal{T})$ is 1-locally strongly compact iff there is a metric d on X such that $(X, \mathcal{T})$ coincides with (X, d) and (X, d) is boundedly strongly compact.

2.3 SIGMA-COMPACT SPACES

2.3.0 theorem 2.2.4 brings us to consider a related but (as we will show) much weaker concept than '1-locally compact'.

DEFINITION: a topological space $(X, \mathcal{T})$ is *sigma-compact* iff it is Hausdorff and there is a sequence $((W_n, \mathcal{T}_{W_n}))_{n \in \mathbb{N}}$ of compact subspaces such that $X = \bigcup_{n \in \mathbb{N}} W_n$.

PROPOSITION: a space $(X, \mathcal{T})$ is sigma-compact iff there is a sequence $(\tau_n)_{n \in \mathbb{N}}$ of fans and a Hausdorff topology $\mathcal{T}'$ on $\sigma = \dot{\bigcup}_{n \in \mathbb{N}} \tau_n$ such that $(X, \mathcal{T})$ coincides with $(\sigma, \mathcal{T}')$.

PROOF: we prove only the non-trivial implication. Let $(X, \mathcal{T})$ be sigma-compact. Let $((W_n, \mathcal{T}_{W_n}))_{n \in \mathbb{N}}$ be a sequence of compact subspaces such that $X = \bigcup_{n \in \mathbb{N}} W_n$. From the proof of theorem 2.2.4 we know that a compact space $(Y, \mathcal{T}')$ can be completely represented by an element $(\tau, \delta_{\approx}, \gamma)$ of $\sigma_\omega \times \sigma_\omega \times \sigma_\omega$, where $(\tau, \#_{\delta_{\approx}})$ is an apartness fan and γ is a homeomorphism from $(\tau, \#_{\delta_{\approx}})$ to $(Y, \mathcal{T}')$. This observation will allow us to apply $\mathbf{AC}_{01}$, since we have:

$(\star)$ $\forall n \in \mathbb{N}\, \exists \tau, \delta_{\approx}, \gamma \in \sigma_\omega\, [\, \gamma$ is a homeomorphism from the apartness fan $(\tau, \#_{\delta_{\approx}})$ to $(W_n, \#)\,]$

By $\mathbf{AC}_{01}$ we obtain a sequence of apartness fans $((\tau_n, \#_n))_{n \in \mathbb{N}}$ and a sequence of spread-functions $(\gamma_n)_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$: γ_n is a homeomorphism from $(\tau_n, \#_n)$ to

$(W_n, \mathcal{T}_{W_n})$. In fact $(\gamma_n)_{n\in\mathbb{N}}$ gives us a surjection γ from $(\biguplus_{n\in\mathbb{N}}\tau_n, d_\omega)=(\sigma, d_\omega)$ to $(X, \mathcal{T})$. The desired Hausdorff topology $\mathcal{T}'$ on σ is simply the collection $\{\gamma^{-1}(U) \mid U\in\mathcal{T}\}$ •

EXAMPLE: we give an example of a sigma-compact metric space (X, d) which is NOT an apartness space. Let $X=\{0\}\cup\mathbb{R}^+$ and let $d=d_{\mathbb{R}}$. For all x in X we can decide: $x=0$ or $x\#0$, so $\{0\}$ is open in $(X, \#)$. But $\{0\}$ is NOT open in (X, d). This also shows that (X, d) is NOT locally compact. On the other hand $(X, \#)$ is locally compact and homeomorphic to $(\{-1\}\cup\mathbb{R}^+, d_{\mathbb{R}})$.

2.3.1 PROPOSITION: every sigma-compact space is weakly metrizable.

PROOF: by the previous proposition 2.3.0 and the definition of 'weakly metrizable' (1.1.3) it suffices to prove the following. Let $(\sigma, \#)=(\biguplus_{n\in\mathbb{N}}\tau_n, \#)$ be an apartness spread, where τ_n is a fan for each $n\in\mathbb{N}$. Then $(\sigma, \#)$ is weakly metrizable. So for $n\in\mathbb{N}$ put $\rho_n=\biguplus_{i\leq n}\tau_n$. Let $n\in\mathbb{N}$ and consider $(\rho_n, \#)$, the apartness space arising from restricting $\#$ to ρ_n. By theorem 2.1.5 there is a metric d on ρ_n metrizing $(\rho_n, \#)$. By lemma 0.4.5 the completion $\overline{(\rho_n, d)}$ is strongly compact. So we find in fact a complete metric fan (ρ, d) such that $(\rho_n, \#)$ coincides with a dense subspace of (ρ, d). By lemma 3.0.3 (using $\mathbf{AC}_{10}$) without loss of generality d is a spread-function from $\rho\times\rho$ to $\mathbb{R}_{\geq 0}$, and so an element of σ_ω. We find:

$(\star)$ $\forall n\in\mathbb{N}\, \exists \tilde{\rho}, d\in\sigma_\omega$ [ρ is a fan and d is a metric on ρ such that (ρ, d) is complete and $(\rho_n, \#)$ coincides with a dense subspace of (ρ, d)]

By $\mathbf{AC}_{01}$ there is a sequence $(\tilde{\rho}_n, d_n)_{n\in\mathbb{N}}$ in $\sigma_\omega\times\sigma_\omega$ realizing $(\star)$. Notice that (ρ_n, d_{n+1}) coincides identically with (ρ_n, d_n) and so $\overline{(\rho_n, d_{n+1})}$ coincides identically with $\overline{(\rho_n, d_n)}$. Therefore without loss of generality $\tilde{\rho}_n\subseteq\tilde{\rho}_{n+1}$. Moreover $(\tilde{\rho}_n, d_{n+1})$ is strongly compact, therefore $(\tilde{\rho}_n, d_{n+1})$ is strongly located in $(\tilde{\rho}_{n+1}, d_{n+1})$ by lemma 0.4.3 combined with corollary 3.2.9.

$\boxed{\text{claim}}$ let a, b be in $\overline{\overline{\sigma}}$ such that $a \not\approx b$. Then there is a spread-function f from $(\sigma, \#)$ to $([0,1], d_{\mathbb{R}})$ such that $f\restriction_{\sigma\cap a}\equiv_{\mathbb{R}}0$ and $f\restriction_{\sigma\cap b}\equiv_{\mathbb{R}}1$.

$\boxed{\text{proof}}$ determine $n\in\mathbb{N}$ such that a, b are in $\overline{\overline{\rho}}_n$. Use the claim in the proof of theorem 2.1.5 to find a continuous spread-function g from (ρ_n, d_n) to $([0,1], d_{\mathbb{R}})$ such that $g\restriction_{\rho_n\cap a}\equiv_{\mathbb{R}}0$ and $g\restriction_{\rho_n\cap b}\equiv_{\mathbb{R}}1$. Clearly g is uniformly continuous. Therefore g can be extended to a continuous function g' from $(\tilde{\rho}_n, d_n)$ to $([0,1], d_{\mathbb{R}})$ using corollary 0.4.2. By $\mathbf{AC}_{11}$ without loss of generality g' is a spread-function.

Let $m \in \mathbb{N}$ and suppose: γ is a continuous spread-function from $(\tilde{\rho}_{n+m}, d_{n+m})$ to $([0,1], d_{\mathbb{R}})$. Since $(\tilde{\rho}_{n+m}, d_{n+m+1})$ is strongly located in $(\tilde{\rho}_{n+m+1}, d_{n+m+1})$, by theorem 4.1.1 there is a continuous extension $\tilde{\gamma}$ of γ, from $(\tilde{\rho}_{n+m+1}, d_{n+m+1})$ to $([0,1], d_{\mathbb{R}})$. By $\mathbf{AC}_{11}$ without loss of generality $\tilde{\gamma}$ is a spread-function. Define for each $m \in \mathbb{N}$ a subset A_m of σ_ω putting $A_m = \{\gamma \mid \gamma$ is a continuous spread-function from $(\tilde{\rho}_{n+m}, d_{n+m})$ to $([0,1], d_{\mathbb{R}})\}$. Put $A = \bigcup_{m \in \mathbb{N}} A_m$. Let R be the subset of $A \times A$ given by: $R = \{(\gamma, \delta) \in A_m \times A_{m+1} \mid \delta$ restricted to ρ_{n+m} equals $\gamma \mid m \in \mathbb{N}\}$. We find:

$(\star)$ $\quad \forall \alpha \in A\ \exists \beta \in A\ [(\alpha, \beta) \in R]$

Since g' is in A_0, by $\mathbf{DC}_1$ we find a sequence $(f_m)_{m \in \mathbb{N}}$ of continuous spread-functions such that $f_0 = g'$ and for each $m \in \mathbb{N}$: (f_m, f_{m+1}) is in R. Then clearly for all $m \in \mathbb{N}$: $f_m \mid_{\rho_n \cap a} \equiv_{\mathbb{R}} 0$ and $f_m \mid_{\rho_n \cap b} \equiv_{\mathbb{R}} 1$.

We define a spread-function f from $(\sigma, \#)$ to $([0,1], d_{\mathbb{R}})$ as follows. First put $f(\alpha) = f_0(\alpha)$ for α in ρ_n. Then for $m \in \mathbb{N}$ and $\alpha \in \{\lessdot n+m \gtrdot \star \beta \mid \beta \in \tau_{n+m}\}$ put $f(\alpha) = f_m(\alpha)$. Clearly f satisfies the claim $\circ$

So $(\sigma, \#)$ satisfies the requirements of the Urysohn lemma 2.1.1, and therefore is weakly metrizable $\bullet$

2.3.2 PROPOSITION: NOT every sigma-compact apartness space is metrizable.

PROOF: let α be in σ_2, then α determines a Σ^1_0-apartness on $\mathbb{N}$ in the following way. Let n, m be in $\mathbb{N}_{\geq 1}$. First put $n \#_\alpha 0$ iff $\exists m \in \mathbb{N}[\alpha_{[n]}(m) = 1]$. Then put $n \#_\alpha m$ iff $n \neq m$ and: $n \#_\alpha 0$ or $m \#_\alpha 0$. For example $(\mathbb{N}, \#_{\underline{0}})$ coincides with $(\{\underline{0}\}, \#_\omega)$ since for all n, m in $\mathbb{N}$: $n \equiv_{\underline{0}} m$. Notice that for all α in σ_2: $(\mathbb{N}, \#_\alpha)$ is sigma-compact. Suppose each sigma-compact space is metrizable. Then in particular $(\mathbb{N}, \#_\alpha)$ is metrizable for all α in σ_2. This gives us:

$(\star)$ $\quad \forall \alpha \in \sigma_2\ \exists \beta \in \sigma_\omega\ [\beta$ codes a metric on $\mathbb{N}$ metrizing $(\mathbb{N}, \#_\alpha)]$

By $\mathbf{AC}_{11}$ there is a spread-function γ in σ_ω realizing $(\star)$. Write d_α for the metric on $\mathbb{N}$ which is coded by $\gamma(\alpha)$, and which metrizes $(\mathbb{N}, \#_\alpha)$. Notice that γ gives us a sequence $(\gamma_n)_{n \in \mathbb{N}}$ of spread-functions from σ_2 to $\mathbb{R}_{\geq 0}$ such that for each $n \in \mathbb{N}$ and $\alpha \in \sigma_2$: $\gamma_n(\alpha) \equiv d_\alpha(0, n)$. We will need the functions $(h_n)_{n \in \mathbb{N}}$ from $\mathbb{N}$ to $\mathbb{N}$ defined by: $h_n(m) = \mu t \in \mathbb{N}\, [t > m \wedge (t)_0 = n]$, for n, m in $\mathbb{N}$.

Now $\gamma_1(\underline{0}) = d_{\underline{0}}(0,1) \equiv_{\mathbb{R}} 0$ so there is a canonical $m_1 \in \mathbb{N}$ such that for all α in σ_2:

$\overline{\alpha}(m_1)=\overline{\underline{0}}(m_1)$ implies $\gamma_1(\alpha)=d_\alpha(0,1)\leq 1$. For we can take m_1 equal to the smallest $k\in\mathbb{N}$ for which $\gamma_1(\overline{\underline{0}}(k))-1\not\approx_{\mathbb{R}}\overline{1_{\mathbb{R}}}(lg(\gamma_1(\overline{\underline{0}}(k))-1))$. Put $\alpha_1=\overline{\underline{0}}(h_1(m_1))\star\lessdot 1\gtrdot\star\underline{0}$. Then α_1 is in σ_2 and $0\,\#_{\alpha_1}1$, but $0\equiv_{\alpha_1}2$. This means $\gamma_2(\alpha_1)=d_{\alpha_1}(0,2)\equiv_{\mathbb{R}}0$. Therefore there is a canonical $m_2\in\mathbb{N}$ such that $m_2>h_1(m_1)$ and for all α in σ_2: $\overline{\alpha}(m_2)=\overline{\alpha}_1(m_2)$ implies $\gamma_2(\alpha)=d_\alpha(0,2)\leq 2^{-1}$ (see the reasoning above). Now put $\alpha_2=\overline{\alpha}_1(h_2(m_2))\star\lessdot 1\gtrdot\star\underline{0}\in\sigma_2$. We find a canonical $m_3\in\mathbb{N}$ such that $m_3>h_2(m_2)$ and for all α in σ_2: $\overline{\alpha}(m_3)=\overline{\alpha}_2(m_3)$ implies $\gamma_3(\alpha)=d_\alpha(0,3)\leq 2^{-2}$.

Continuing in this fashion, using (only) $\mathbf{AC}_{00}$ we find a sequence $(m_n)_{n\in\mathbb{N}_{\geq 1}}$ in $\mathbb{N}$ and a sequence $(\alpha_n)_{n\in\mathbb{N}_{\geq 1}}$ in σ_2 such that $\alpha=d_\omega\text{-}\lim(\alpha_n)_{n\in\mathbb{N}_{\geq 1}}$ is an element of σ_2, and moreover: $\forall n\in\mathbb{N}_{\geq 1}\,[\,0\,\#_\alpha n\wedge d_\alpha(0,n)\leq 2^{-n+1}\,]$. But then clearly d_α does not metrize $(\mathbb{N},\#_\alpha)$ since $\{0\}$ is open in in $(\mathbb{N},\#_\alpha)$, but not in $(\mathbb{N},d_\alpha)$. Contradiction •

REMARK: this shows that NOT every sigma-compact apartness space is 1-locally compact. There also exists a *metrizable* apartness space which is sigma-compact but NOT locally compact, see 3.3.14.

2.3.3 we end this section with a remark on sigma-compact apartness spaces. Let $(X,\mathcal{T})$ be a sigma-compact apartness space. Then $(X,\mathcal{T})$ coincides with an apartness spread $(\sigma,\#)$ with $\sigma=\biguplus_{n\in\mathbb{N}}\tau_n$, where $(\tau_n)_{n\in\mathbb{N}}$ is a sequence of fans. For $n\in\mathbb{N}$ put $\rho_n=\uplus_{i\leq n}\tau_n$, then $((\rho_n,\#))_{n\in\mathbb{N}}$ is an increasing sequence of compact subspaces of $(\sigma,\#)$. Notice that a subset U of σ is open in $(\sigma,\#)$ iff for all $n\in\mathbb{N}$: $U\cap\rho_n$ is open in $(\rho_n,\#)$.

On the other hand, let $(X,\mathcal{T})$ be a topological space and $((W_n,\mathcal{T}_{W_n}))_{n\in\mathbb{N}}$ is an increasing sequence of compact subspaces of $(X,\mathcal{T})$ such that: $X=\bigcup_{n\in\mathbb{N}}W_n$ and a subset U of X is in $\mathcal{T}$ iff for all $n\in\mathbb{N}$: $U\cap W_n$ is in $\mathcal{T}_{W_n}$. Then it is easy to see that $(X,\mathcal{T})$ is a sigma-compact apartness space.

Knowing classical topology we might therefore say: a topological space $(X,\mathcal{T})$ is a sigma-compact apartness space iff it is the *inductive limit* of an increasing sequence of compact subspaces.

2.4 STAR-FINITARY SPACES

2.4.0* in this section we investigate star-finite apartness spreads, which form a natural generalization of locally compact spreads.

DEFINITION: let σ be a spread and let $\approx$ be a touch-relation on $\overline{\overline{\sigma}}$. Then $\approx$ is called *star-finite* iff for all a in $\overline{\overline{\sigma}}$ the set $\{b\in\overline{\sigma}(lg(a))\,|\,a\approx b\}$ is finite. Now let $\#$ be an apartness on σ. Then $(\sigma,\#)$ is called *star-finite* iff there is a star-finite touch-relation corresponding to $\#$. Finally we say that a topological space $(X,\mathcal{T})$ is *star-finitary* iff $(X,\mathcal{T})$ coincides with a star-finite apartness spread.

The two examples of star-finite apartness spreads which prompted this definition are $(\sigma_\omega,\,\#_\omega)$ and $(\mathbb{R}^{\mathbb{N}},\,\#_{\mathbb{R}^{\mathbb{N}}})$. Also, any 1-locally compact space $(X,\mathcal{T})$ is star-finitary. And it is not difficult to see that if $(X_n,\mathcal{T}_n)_{n\in\mathbb{N}}$ is a sequence of star-finitary spaces, then the topological product $(\Pi_{n\in\mathbb{N}}X_n,\mathcal{T}_{\mathrm{prod}})$ (see 1.0.9) is star-finitary. This shows that the class of star-finite apartness spreads is quite large. Notice that a star-finitary $(\sigma,\#)$ need not be star-finite; a simple example is $(\sigma_\omega,\emptyset)$ where $\emptyset$ is the empty apartness on σ_ω (all elements are equivalent).

2.4.1* we wish to show that a star-finite $(\sigma,\#)$ is metrizable. For this we will have to generalize our method for metrizing an apartness fan $(\tau,\#)$, see section 2.1. The key to all our results is the simple observation that in a star-finite $(\sigma,\#)$, for each α in σ the equivalence class of α, that is $\{\beta\in\sigma\,|\,\beta\equiv\alpha\}$ is contained in a subfan of σ.

DEFINITION: let $(\sigma,\#)$ be a star-finite apartness spread with corresponding star-finite touch-relation $\approx$ on $\overline{\overline{\sigma}}$. Let α be in σ. We inductively define a subfan $\tau_{\alpha,\approx}$ of σ as follows. Put $\tau_{\alpha,\approx}(\lessdot\gtrdot)=0$. Now let a in $\mathbb{N}_{\geq 1}$ and suppose $\tau_{\alpha,\approx}(a')$ has been defined, where $a'=\lessdot a_0,\ldots,a_{lg(a)-2}\gtrdot$. Then:

$$\tau_{\alpha,\approx}(a)\ \underset{D}{=}\ \begin{cases} 0 & \text{if } a\approx\overline{\alpha}(lg(a)) \\ 0 & \text{if } \tau_{\alpha,\approx}(a')=0 \text{ and } \forall b\in\overline{\overline{\sigma}}\ [\,a'\sqsubset b\rightarrow b\not\approx\overline{\alpha}(lg(a))\,] \text{ and} \\ & a_{lg(a)-1}=\mu t\in\mathbb{N}\,[\sigma(a'\star\lessdot t\gtrdot)=0] \\ 1 & \text{else} \end{cases}$$

The definition hinges on the finiteness of $\{b\in\overline{\overline{\sigma}}\,|\,b\approx\overline{\alpha}(n)\}$, for all $n\in\mathbb{N}$. Clearly $\tau_{\alpha,\approx}$ is a subfan of σ, and $\{\beta\in\sigma\,|\,\beta\equiv\alpha\}$ is contained in $\tau_{\alpha,\approx}$.

2.4.2 LEMMA: let $(\sigma,\#)$ and $\approx$ be as above. Let U be open in $(\sigma,\#)$, and let α be in U. Then there is an $N\in\mathbb{N}$ such that for all β in σ: $\overline{\beta}(N)\approx\overline{\alpha}(N)$ implies $\beta\in U$.

PROOF: We have:

$$(\star)\quad \forall\beta\in\sigma\ \exists(s,n)\in\{0,1\}\times\mathbb{N}\ [\,(s=0\wedge\beta\in U)\vee(s=1\wedge\overline{\beta}(n)\not\approx\overline{\alpha}(n))\,]$$

By $\mathbf{AC}_{10}$ there is a spread-function γ from σ to $\{0,1\}\times\mathbb{N}$ realizing $(\star)$. Then we find:

$(\star\star)$ $\quad \forall\delta\in\tau_{\alpha,\approx}\ \exists!n\in\mathbb{N}\ [\,\gamma(\overline{\delta}(n))>0\,]$

By the fan theorem **FT** the set $\{\overline{\delta}(n)\mid n\in\mathbb{N},\delta\in\tau_{\alpha,\approx},\gamma(\overline{\delta}(n))>0\}$ is finite. This gives us an $N\in\mathbb{N}$ such that for all δ in $\tau_{\alpha,\approx}$: $\overline{\delta}(N)\approx\overline{\alpha}(N)$ implies $\exists i\leq N\ [\,\gamma(\overline{\delta}(i))=1\,]$ implies $\delta\in U$. But then by definition of $\tau_{\alpha,\approx}$ for all β in σ: $\overline{\beta}(N)\approx\overline{\alpha}(N)$ implies $\exists i\leq N\ [\,\gamma(\overline{\beta}(i))=1\,]$ implies $\beta\in U$ •

The proof illustrates the usefulness of $\tau_{\alpha,\approx}$. It acts almost as a neighborhood of α in $(\sigma,\#)$. However, in general $\tau_{\alpha,\approx}$ is too small a fan for our purposes. We make a slight detour to nicely introduce a similar but larger fan.

2.4.3 DEFINITION: let $(\sigma,\#)$ be a star-finite apartness spread with corresponding star-finite touch-relation $\approx$ on $\overline{\overline{\sigma}}$. We inductively define for each $n\in\mathbb{N}$ a binary relation $\stackrel{n}{\approx}$ on $\overline{\overline{\sigma}}$ as follows. Let $a\,,b$ in $\overline{\overline{\sigma}}$, with $lg(a)\leq lg(b)$. Then $a\stackrel{0}{\approx}b$ iff $b\stackrel{0}{\approx}a$ iff $a\approx b$. Now for $n\in\mathbb{N}$: $a\stackrel{n+1}{\approx}b$ iff $b\stackrel{n+1}{\approx}a$ iff there is $c\in\overline{\overline{\sigma}}(lg(a))$ such that $a\approx c$ and $c\stackrel{n}{\approx}b$.

LEMMA: let $(\sigma,\#)$ and $\approx$ be as above. Then $\stackrel{1}{\approx}$ is a star-finite touch-relation on $\overline{\overline{\sigma}}$ corresponding to $\#$.

PROOF: clearly $\stackrel{1}{\approx}$ satisfies definition 2.0.2 (i). For said definition (ii) we must verify that $\stackrel{1}{\approx}$ induces an apartness, the apartness $\#$ to be precise. Let $\alpha\,,\beta$ be in σ. If there is an $n\in\mathbb{N}$ such that $\overline{\alpha}(n)\stackrel{1}{\not\approx}\overline{\beta}(n)$ then clearly $\alpha\#\beta$. Now suppose $\alpha\#\beta$, we will show there is an $N\in\mathbb{N}$ such that $\overline{\alpha}(N)\stackrel{1}{\not\approx}\overline{\beta}(N)$. We have:

$(\star)$ $\quad \forall\gamma\in\tau_{\alpha,\approx}\ \exists n\in\mathbb{N}\ [\,\overline{\gamma}(n)\not\approx\overline{\alpha}(n)\vee\overline{\gamma}(n)\not\approx\overline{\beta}(n)\,]$

By the fan theorem **FT** there is an $N\in\mathbb{N}$ such that for all γ in $\tau_{\alpha,\approx}$: $\overline{\gamma}(N)\not\approx\overline{\alpha}(N)\vee\overline{\gamma}(N)\not\approx\overline{\beta}(N)$. Clearly then $\overline{\alpha}(N)\stackrel{1}{\not\approx}\overline{\beta}(N)$. That $\stackrel{1}{\approx}$ is star-finite is trivial •

COROLLARY: for all $n\in\mathbb{N}$: $\stackrel{n}{\approx}$ is a star-finite touch-relation on $\overline{\overline{\sigma}}$ corresponding to $\#$.

Now to generalize our method for metrizing an apartness fan to $(\sigma,\#)$ we will need $\tau_{\alpha,\stackrel{1}{\approx}}$ (which contains $\tau_{\alpha,\approx}$). The next lemma is a generalization of lemma 2.1.3. This lemma says that if $(\tau,\#)$ is an apartness fan, and $A\,,B$ are finite subsets of $\overline{\overline{\tau}}$ such that for all $\alpha\in\tau_A$ and all $\beta\in\tau_B$: $\alpha\#\beta$, then there is an $M\in\mathbb{N}$ such that $\overline{\overline{\tau}}_A(M)\stackrel{2}{\not\approx}\overline{\overline{\tau}}_B(M)$, see also remark 2.1.3. Remember definition 2.0.1, that if σ is a spread and A is a subset of $\overline{\overline{\sigma}}$, we write σ_A for $\bigcup_{a\in A}\sigma\cap a$, as well as $\overline{\overline{\sigma}}_A(n)$ for $\{\overline{\alpha}(n)\,|\,\alpha\in\sigma_A\}$ and $\overline{\overline{\sigma}}_A$ for $\bigcup_{n\in\mathbb{N}}\overline{\overline{\sigma}}_A(n)$.

2.4.4 LEMMA: let $(\sigma,\#)$ be star-finite with corresponding star-finite touch-relation $\approx$ on $\overline{\overline{\sigma}}$. Let A,B be two decidable subsets of $\overline{\overline{\sigma}}$ such that for all $\alpha\in\sigma_A$ and all $\beta\in\sigma_B$: $\alpha\#\beta$. Suppose moreover that for each $\alpha\in\sigma$: $A\cap\overline{\overline{\tau}}_{\alpha,\frac{1}{\approx}}$ and $B\cap\overline{\overline{\tau}}_{\alpha,\frac{1}{\approx}}$ are finite. Then there is a canonical spread-function γ from σ to $\{0,1,2\}$ such that if we put $A_{-1}=A$, and $A_{\lessdot 0\gtrdot}=\{a\in\overline{\overline{\sigma}}\,|\,\gamma(a)=1\}$, $A_{\lessdot 1\gtrdot}=\{a\in\overline{\overline{\sigma}}\,|\,\gamma(a)=2\}$, $A_{\lessdot 2\gtrdot}=\{a\in\overline{\overline{\sigma}}\,|\,\gamma(a)=3\}$, and $A_{\lessdot 3\gtrdot}=B$, then the following holds. If α is in σ and $M\in\mathbb{N}$ is such that $\gamma(\overline{\alpha}(M))>0$ then :

(i) $M\geq\max(\{0\}\cup\{lg(b)\mid b\in(A\cup B)\cap\overline{\overline{\tau}}_{\alpha,\frac{1}{\approx}}\})$

(ii) $\overline{\overline{\sigma}}_A(M)\cap\overline{\overline{\tau}}_{\alpha,\frac{1}{\approx}}\overset{2}{\not\approx}\overline{\overline{\sigma}}_B(M)\cap\overline{\overline{\tau}}_{\alpha,\frac{1}{\approx}}$.

(iii) $\gamma(\overline{\alpha}(M))=1$ implies $\overline{\alpha}(M)\approx\overline{\overline{\sigma}}_A(M)$, $\gamma(\overline{\alpha}(M))=3$ implies $\overline{\alpha}(M)\approx\overline{\overline{\sigma}}_B(M)$ and $\gamma(\overline{\alpha}(M))=2$ implies $\overline{\overline{\sigma}}_A(M)\not\approx\overline{\alpha}(M)\not\approx\overline{\overline{\sigma}}_B(M)$.

(iv) for all i,j in $\{-1,\lessdot 0\gtrdot,\lessdot 1\gtrdot,\lessdot 2\gtrdot,\lessdot 3\gtrdot\}$: $i\notin\{Pred(j),j,Succ(j)\}$ implies $\forall\beta\in\sigma_{A_i}\ \forall\delta\in\sigma_{A_j}\ [\beta\#\delta]$. Moreover $A_i\cap\overline{\overline{\tau}}_{\alpha,\frac{1}{\approx}}$ is finite.

PROOF: let α be in σ. We have: $A\cap\overline{\overline{\tau}}_{\alpha,\frac{1}{\approx}}$ and $B\cap\overline{\overline{\tau}}_{\alpha,\frac{1}{\approx}}$ are finite, and for all β in $A\cap\tau_{\alpha,\frac{1}{\approx}}$ and δ in $B\cap\tau_{\alpha,\frac{1}{\approx}}$: $\beta\#\delta$. So we can apply lemma 2.1.3 to find the smallest $M\in\mathbb{N}$ such that $M\geq\max(\{0\}\cup\{lg(b)\mid b\in(A\cup B)\cap\overline{\overline{\tau}}_{\alpha,\frac{1}{\approx}}\})$ and $A\cap\overline{\overline{\tau}}_{\alpha,\frac{1}{\approx}}(M)\overset{2}{\not\approx}B\cap\overline{\overline{\tau}}_{\alpha,\frac{1}{\approx}}(M)$. Then (i) and (ii) are satisfied, so we can define $\gamma(\overline{\alpha}(M))$ such that (iii) is satisfied. Notice that since α is arbitrary, this completely determines a canonical spread-function γ from σ to $\{0,1,2\}$.

We must prove (iv) for this spread-function γ. Let β,δ be in σ, and determine $N_0,N_1\in\mathbb{N}$ such that $\gamma(\overline{\beta}(N_0))>0$ and $\gamma(\overline{\delta}(N_1))>0$. Suppose $\gamma(\overline{\beta}(N_0))=2$, that is: $\beta\in\sigma_{A_{\lessdot 1\gtrdot}}$, and suppose $\delta\in\sigma_A\cup\sigma_B=\sigma_{A_{-1}}\cup\sigma_{A_{\lessdot 3\gtrdot}}$. Then by definition of γ: $\overline{\beta}(N_0)\not\approx\overline{\delta}(N_0)$ so $\beta\#\delta$. Now suppose $\beta\in\sigma_{A_{\lessdot 0\gtrdot}}$ and $\delta\in\sigma_{A_{\lessdot 2\gtrdot}}$. By symmetry we may assume, without loss of generality, that $N_0\leq N_1$. Since $\overline{\delta}(N_1)\approx\overline{\overline{\sigma}}_B(N_1)$ we find: $\overline{\delta}(N_0)\approx\overline{\overline{\sigma}}_B(N_0)$. Let c be in $\overline{\overline{\sigma}}_B(N_0)$ such that $\overline{\delta}(N_0)\approx c$. Suppose $\overline{\beta}(N_0)\approx\overline{\delta}(N_0)$. Then $\overline{\beta}(N_0)\approx\overline{\delta}(N_0)\approx c$, so c is in $\overline{\overline{\tau}}_{\beta,\frac{1}{\approx}}(N_0)$. We find: $\overline{\overline{\sigma}}_A(N_0)\cap\overline{\overline{\tau}}_{\beta,\frac{1}{\approx}}\approx\overline{\beta}(N_0)\approx\overline{\delta}(N_0)\approx\overline{\overline{\sigma}}_B(N_0)\cap\overline{\overline{\tau}}_{\beta,\frac{1}{\approx}}$. But this contradicts (i) and the definition of γ. Therefore $\overline{\beta}(N_0)\not\approx\overline{\delta}(N_0)$ meaning $\beta\#\delta$.

Finally, that $A_i\cap\overline{\overline{\tau}}_{\alpha,\frac{1}{\approx}}$ is finite for all i in $\{-1,\lessdot 0\gtrdot,\lessdot 1\gtrdot,\lessdot 2\gtrdot,\lessdot 3\gtrdot\}$ follows for i in $\{-1,\lessdot 3\gtrdot\}$ by assumption. Now let i in $\{\lessdot 0\gtrdot,\lessdot 1\gtrdot,\lessdot 2\gtrdot\}$. Since γ is a spread-function, we have:

(★) $\forall \beta \in \tau_{\alpha,\frac{1}{\approx}}\ \exists! n \in \mathbb{N}\ [\gamma(\overline{\beta}(n)) > 0]$

The fan theorem **FT** now implies that $A_i \cap \overline{\overline{\tau}}_{\alpha,\frac{1}{\approx}}$ is finite •

2.4.5 we will use the previous lemma in much the same way as lemma 2.1.3 is used in proving an apartness fan metrizable (theorem 2.1.5). That is, we apply it repeatedly in order to obtain, for a, b in $\overline{\overline{\sigma}}$ such that $a \not\approx b$, a spread-function $\gamma_{a,b}$ from σ to $([0,1], d_{\mathbb{R}})$ such that $\gamma_{a,b} \mid_{\sigma \cap a} \equiv_{\mathbb{R}} 0$ and $\gamma_{a,b} \mid_{\sigma \cap b} \equiv_{\mathbb{R}} 1$. Then by the Urysohn lemma 2.1.1 $(\sigma, \#)$ is weakly metrizable. But the metric constructed in lemma 2.1.1 need not metrize $(\sigma, \#)$, unless we take some special precautions. Therefore we first expand $(\sigma, \#)$ with a single isolated point, which without loss of generality we can take to be $\underline{0}$. So we put $\overline{\underline{0}}(n+1) \not\approx a$ for all $n \in \mathbb{N}$ and a in $\overline{\overline{\sigma}}$, $lg(a) \geq 1$. The resulting expanded spread is again star-finite. Then using lemma 2.4.4 we construct $\gamma_{a,b}$ for a in $\overline{\overline{\sigma}}$, $lg(a) \geq 1$ and $b = \overline{\underline{0}}(lg(a))$. Actually $\gamma_{a,\overline{\underline{0}}(lg(a))}$ is a sequence of spread-functions $(\gamma_n)_{n \in \mathbb{N}}$ from $(\sigma, \#)$ to $\{0,1,2\}$. We took special care in formulating lemma 2.4.4, especially (i) of its conclusion. Thus we now obtain that if a, c are in $\overline{\overline{\sigma}}(lg(a))$ and $a \not\approx c$, then for all β in $\sigma \cap c$: $\gamma_{a,\overline{\underline{0}}(lg(a))}(\beta) \geq \frac{1}{3}$. In this way we ensure that the constructed weakly metrizing metric will indeed metrize $(\sigma, \#)$.

LEMMA: let $(\sigma, \#)$ be a star-finite apartness spread, with corresponding star-finite touch-relation $\approx$ on $\overline{\overline{\sigma}}$. Let d be a metric on σ which weakly metrizes $(\sigma, \#)$, and such that if a is in $\overline{\overline{\sigma}}$, then there is an ϵ in $\mathbb{R}^+$ such that for all c in $\overline{\overline{\sigma}}(lg(a))$, $c \not\approx a$, for all α in $\sigma \cap a$ and all β in $\sigma \cap c$: $d(\alpha, \beta) > \epsilon$. Then d metrizes $(\sigma, \#)$.

PROOF: it is easy to see that $(\sigma, \#)$ refines (σ, d). Now let U be open in $(\sigma, \#)$, we show that U is open in (σ, d). Let α in U. By lemma 2.4.2 there is an $N \in \mathbb{N}$ such that for all β in σ: $\overline{\beta}(N) \approx \overline{\alpha}(N)$ implies $\beta \in U$. On the other hand, by assumption there is ϵ in $\mathbb{R}^+$ such that if β is in σ and $\overline{\beta}(N) \not\approx \overline{\alpha}(N)$, then $d(\alpha, \beta) > \epsilon$. Put these two observations together to obtain that $B(\alpha, \epsilon) \subseteq U$ •

2.4.6 THEOREM: every star-finitary space is metrizable.

PROOF: it suffices to prove that a given star-finite apartness spread $(\sigma, \#)$ is metrizable. We first expand $(\sigma, \#)$ with a single isolated point. Without loss of generality $\langle\!\langle 0 \rangle\!\rangle$ is not in $\overline{\overline{\sigma}}$. Put $\rho = \sigma \cup \{\underline{0}\}$ and expand $\approx$ to $\overline{\overline{\rho}}$ putting $\overline{\underline{0}}(n+1) \not\approx a$ for all $n \in \mathbb{N}$ and a in $\overline{\overline{\sigma}}$, $lg(a) \geq 1$. Then $(\rho, \#)$ is star-finite, with corresponding star-finite $\approx$. Let c, e be in $\overline{\overline{\rho}}$ such that $c \not\approx e$.

$\boxed{\text{claim}}$ there is a canonical spread-function γ from $(\rho, \#)$ to $([0,1]_3, d_{\mathbb{R}})$ such that $\gamma \mid_{\rho\cap c} \equiv_{\mathbb{R}} 0$ and $\gamma \mid_{\rho\cap e} \equiv_{\mathbb{R}} 1$. Moreover if c' is in $\overline{\overline{\rho}}(lg(c))$ such that $c \not\approx c'$ then $\gamma(\beta) \geq \frac{1}{3}$ for all β in $\rho\cap c'$.

$\boxed{\text{proof}}$ put $A_0 = \overline{\overline{\rho}}$, $A_{-1} = \{c\}$ and $A_{\lessdot 3 \gtrdot} = \{e\}$. With induction we define, for each $n \in \mathbb{N}_{\geq 1}$, a canonical spread-function γ_n from ρ to $\overline{\overline{\sigma}}_3(n)$ such that if we put $A_a = \{p \in \overline{\overline{\rho}} \mid \gamma_n(p) = a+1\}$, for $a \in \overline{\overline{\sigma}}_3(n)$, then:

(i) $\forall a, b \in \overline{\overline{\sigma}}_3(n) \cup \{-1, \lessdot 3 \gtrdot\}\ [a \notin \{Pred(b), b, Succ(b)\} \to \forall \alpha \in \rho_{A_a}\ \forall \beta \in \rho_{A_b}\ [\alpha \# \beta]]$.

(ii) $A_{a \star \lessdot i \gtrdot} \subseteq A_a$ for $a \in \overline{\overline{\sigma}}_3(n-1)$ and $i \in \{0,1,2\}$.

Basis: $n=1$. Clearly A_{-1} and $A_{\lessdot 3 \gtrdot}$ are two decidable subsets of $\overline{\overline{\rho}}$ meeting the requirements of lemma 2.4.4. So we use this lemma to find a canonical spread-function δ_0 from ρ to $\{0,1,2\}$ realizing the conclusion of lemma 2.4.4. It is easy to see that (i) and (ii) are satisfied.

Induction: let $n \in \mathbb{N}_{\geq 1}$ and suppose γ_n has been defined and satisfies (i) and (ii) above. Let a be in $\overline{\overline{\sigma}}_3(n)$. Consider $A_{Pred(a)}$ and $A_{Succ(a)}$. Since γ_n is a spread-function, we find for α in ρ that $A_{Pred(a)} \cap \tau_{\alpha,\approx}$ and $A_{Succ(a)} \cap \tau_{\alpha,\approx}$ are finite, using **FT** (see the end of the proof of lemma 2.4.4). Together with (i) this implies that $A_{Pred(a)}$ and $A_{Succ(a)}$ satisfy the conditions of lemma 2.4.4. So we find a canonical spread-function δ_a from ρ to $\{0,1,2\}$ realizing the conclusion of lemma 2.4.4 (for $A_{Pred(a)}$ and $A_{Succ(a)}$). Now let α be in ρ. We define γ_{n+1} by putting: $\gamma_{n+1}(\alpha) = \gamma_n)(\alpha) \star \lessdot \delta_{\gamma_n(\alpha)}(\alpha) \gtrdot$, and stipulating for $N \in \mathbb{N}$: $\gamma_{n+1}(\overline{\alpha}(N)) > 0$ iff $N = \mu t \in \mathbb{N}\,[\exists i, j \leq t[\delta_{\gamma_n(\alpha)}(\overline{\alpha}(i)) > 0 \wedge \gamma_n(\overline{\alpha}(j)) > 0]]$. It is not difficult to see that (i) and (ii) are now satisfied by γ_{n+1}.

Now define a trivial spread-function γ_0 from ρ to $\overline{\overline{\sigma}}_3(0)$ putting $\gamma_0(\alpha) = \lessdot\gtrdot$. All together this gives us a special sequence $(\gamma_n)_{n\in\mathbb{N}}$ of spread-functions from ρ to $\overline{\overline{\sigma}}_3$. We define the promised spread-function γ from ρ to $([0,1]_3, d_{\mathbb{R}})$ by putting: $\overline{\gamma(\alpha)}(n) = \gamma_n(\alpha)$, for α in ρ and $n \in \mathbb{N}$. It is not difficult to verify that γ is a spread-function from $(\rho, \#)$ to $([0,1]_3, d_{\mathbb{R}})$ such that $\gamma \mid_{\rho\cap c} \equiv_{\mathbb{R}} 0$ and $\gamma \mid_{\rho\cap e} \equiv_{\mathbb{R}} 1$. To finish the proof, let c' be in $\overline{\overline{\rho}}(lg(c))$ such that $c \not\approx c'$, and let β in $\rho\cap c'$. We hold: $\gamma_1(\beta) \in \{1,2\}$. This follows from lemma 2.4.4 (i) and (iii). For suppose to the contrary that $\gamma_1(\beta) = 0$. Then if $M \in \mathbb{N}$ is such that $\gamma_1(\overline{\beta}(M) = 1$ we see by lemma 2.4.4 (iii) that $\overline{\beta}(M) \approx \overline{\overline{\sigma}}_{A_{-1}}(M)$. Clearly this implies that $c \in \tau_{\beta,\frac{1}{\approx}}$. But then $M \geq lg(c)$, and so $c' \sqsubseteq \overline{\beta}(M)$. But $c' \not\approx c$ so $c' \not\approx \overline{\overline{\sigma}}_{A_{-1}}(M)$, so $\overline{\beta}(M) \approx \overline{\overline{\sigma}}_{A_{-1}}(M)$. Contradiction. Therefore $\gamma_1(\beta) \in \{1,2\}$, and this means $\gamma(\beta) \geq \frac{1}{3}$ ∘

Since the γ in the claim can be found canonically, we obtain a sequence $(\gamma_{a,b})_{a,b\in\overline{\overline{\rho}}}$ of spread-functions from $(\rho, \#)$ to $([0,1]_3, d_{\mathbb{R}})$, such that for a, b in $\overline{\overline{\rho}}$: $a \approx b$ implies $\gamma_{a,b} \equiv_{\mathbb{R}} 0$, whereas $a \not\approx b$ implies $\gamma_{a,b} \mid_{\sigma\cap a} \equiv_{\mathbb{R}} 0$ and $\gamma_{a,b} \mid_{\sigma\cap b} \equiv_{\mathbb{R}} 1$. Let h

be an enumeration of $\overline{\overline{\rho}}\times\overline{\overline{\rho}}$. Define a metric d on ρ by putting, for α,β in ρ: $d(\alpha,\beta)=\sum_{n\in\mathbb{N}}2^{-n}\cdot|\gamma_{h(n)}(\alpha)-\gamma_{h(n)}(\beta)|$.

$\boxed{\text{claim}}$ let a be in $\overline{\overline{\rho}}$, $lg(a)\geq 1$. Then there is an ϵ in $\mathbb{R}^+$ such that if b is in $\overline{\overline{\rho}}(lg(a))$ such that $a\not\approx b$, then for all α in $\rho\cap a$ and all β in $\rho\cap b$: $d(\alpha,\beta)>\epsilon$.

$\boxed{\text{proof}}$ since we constructed $(\rho,\#)$ by expanding $(\sigma,\#)$ with a single isolated point, there surely is a c in $\overline{\overline{\rho}}(lg(a))$ such that $a\not\approx c$. Now let b be in $\overline{\overline{\rho}}(lg(a))$ such that $a\not\approx b$, and let β in $\rho\cap b$. By the previous claim we see that $\gamma_{a,c}(\beta)\geq\frac{1}{3}$. On the other hand, for all α in $\rho\cap a$ $\gamma_{a,c}(\alpha)\equiv_{\mathbb{R}}0$. So we can take $\epsilon=2^{-h^{-1}(a,c)-1}\cdot\frac{1}{3}$ $\circ$

By lemma 2.4.5 we see that d metrizes $(\rho,\#)$. Clearly σ is open in $(\rho,\#)$, so by lemma 1.3.1 (σ,d) coincides with $(\sigma,\#)$, meaning d metrizes $(\sigma,\#)$ $\bullet$

2.4.7 we end this section with a less important definition:

DEFINITION: let let σ be a spread and let $\approx$ be a touch-relation on $\overline{\overline{\sigma}}$. Then $\approx$ is called *weakly star-finite* iff for all a in $\overline{\overline{\sigma}}$ there is an $M\in\mathbb{N}$ such that the set $\{b\in\overline{\overline{\sigma}}(lg(a))\,|\,a\approx b\}$ contains at most M elements. Now let $\#$ be an apartness on σ. Then $(\sigma,\#)$ is called *weakly star-finite* iff there is a weakly star-finite touch-relation corresponding to $\#$.

The reason for this definition, which classically would coincide with the definition of 'star-finite' (def. 2.4.0), is that every complete metric space coincides with a weakly star-finite apartness spread. Such structure theorems are rare, due to the broadness of the concept 'complete metric space'. We also prove that NOT every complete metric space coincides with a star-finite apartness spread. We refer the reader to theorems 3.1.8 and 3.1.9. Another structure theorem for metric spaces is the following lemma.

LEMMA: let (X,d) be a metric space. Then (X,d) can be embedded in the Hilbert cube $(\mathcal{Q},d_{\mathcal{Q}})$.

PROOF: let $(x_n)_{n\in\mathbb{N}}$ be dense in (X,d). Define a function f from (X,d) to $(\mathcal{Q},d_{\mathcal{Q}})$ by putting, for x in X: $f(x)(n)=\frac{d(x,x_n)}{1+d(x,x_n)}$. It is easy to verify that f is an embedding of (X,d) in $(\mathcal{Q},d_{\mathcal{Q}})$ $\bullet$

In other words, the Hilbert cube is a compact extension of every metric space.

CHAPTER THREE

METRIC TOPOLOGY

abstract

If (X, d) is a metric space and $\mathcal{U}$ is a *per-enumerable* cover of (X, d), then there is a star-finite refinement $\mathcal{V}$ of $\mathcal{U}$, and also a partition of unity $(p_n)_{n\in\mathbb{N}}$ subordinate to $\mathcal{U}$. This theorem holds already in Bishop's school. Using only $\mathbf{AC}_{10}$, we show that an arbitrary open cover of a metric spread (σ, d) has a per-enumerable refinement. Consequently every metric spread is normal. Every complete metric space is spreadlike. In fact every complete metric space coincides with a weakly star-finite apartness spread. NOT every complete metric space coincides with a star-finite apartness spread. Introduction of new concepts of locatedness of subsets in a metric space (X, d). Introduction of 'weakly stable', a topological property that a metric space can possess. Each metric space (X, d) has a weakly stable closure $\overline{(X, d)}$, and the weakly stable closure of a metric spread is again spreadlike. Each complete metric space is weakly stable. Using $\mathbf{AC}_{10}$ we obtain a generalization of the Continuity Principle $\mathbf{CP}$, called Weakly Stable Continuity Principle ($\mathbf{CP}_{\mathrm{ws}}$). As a consequence we obtain that in a weakly stable metric spread (σ, d) the metric topology coincides with the $\#_d$-topology. This in turn implies that everywhere defined functions from a weakly stable metric spread to another metric space are metrically continuous.

3.0 EVERY COMPLETE METRIC SPACE IS SPREADLIKE

3.0.0* for the convenience of the reader we partly repeat ourselves:

DEFINITION: let (X,d) be a metric space, let A be a subset of (X,d). We say that A is *open* in (X,d) iff for all x in U there is an $n\in\mathbb{N}$ such that $B(x,2^{-n})\subseteq U$. A is *closed* in (X,d) iff any d-Cauchy-sequence $(a_n)_{n\in\mathbb{N}}$ in A which converges in (X,d), the limit $d\text{-lim}(a_n)_{n\in\mathbb{N}}$ is in A.

REMARK: (A,d) is complete iff A is closed in any space (X,d) which contains (A,d) as a subspace. Notice that the complement of an open set in (X,d) is closed in (X,d). The implication the other way round is problematic.

3.0.1* we will need the following technical definition.

DEFINITION: let (σ,d) be a metric spread. Then (σ,d) is called *steady* iff for all a in $\overline{\overline{\sigma}}$: $lg(a)>0$ implies $\forall\alpha,\beta\in\sigma\cap a\ [d(\alpha,\beta)<2^{-lg(a)}]$.

REMARK: every enumerable space (X,d) coincides with a steady metric spread. For let $X=\{x_n \mid n\in\mathbb{N}\}$. Then clearly there is a surjection from $\sigma_{\mathbb{N}}$ to (X,d). Defining d on $\sigma_{\mathbb{N}}$ in the obvious way we trivially have: $(\sigma_{\mathbb{N}},d)$ is steady and coincides with (X,d).

3.0.2* THEOREM: let (X,d) be a metric space. Then $\overline{(X,d)}$ coincides with a steady metric spread (σ,d).

PROOF: let $(x_n)_{n\in\mathbb{N}}$ be dense in (X,d). We have:

$$(\star)\quad \forall(n,m,t,p)\in\mathbb{N}^4\ \exists s\in\{0,1\}\ [\,(s=0\wedge d(x_n,x_m)<2^{-t})\vee(s=1\wedge d(x_n,x_m)>2^{-t}-2^{-p})\,]$$

By $\mathbf{AC}_{00}$ there is a function h from $\mathbb{N}^4$ to $\mathbb{N}$ realizing $(\star)$. Define a spread σ as follows. Let a be in $\mathbb{N}$, then:

$$\sigma(a) \mathrel{\overline{\overline{{}_D}}} \begin{cases} 0 & \text{if } a_0\neq\lessdot\gtrdot \text{ and } \forall i<lg(a)\,\forall j<i\,[h((a_j)_0,(a_i)_0,j+2,(a_i)_1)=0] \\ 1 & \text{else} \end{cases}$$

Trivially $\sigma(\lessdot\gtrdot)=0$. If a is in $\mathbb{N}$, $lg(a)\geq 1$, such that $\sigma(a)=0$, then $\sigma(a\star\lessdot a_{lg(a)-1}\gtrdot)=0$. Therefore σ is a spread. Observe that an α in σ codes a Cauchy-sequence in (X,d), namely $(x_{(\alpha(n))_0})_{n\in\mathbb{N}}$, with the property that for all $n,m\in\mathbb{N},\ n\leq m:\ d(x_{(\alpha(n))_0},x_{(\alpha(m))_0})<2^{-n-2}$. On the other hand, if β is in σ_ω such that for all $n,m\in\mathbb{N},\ n\leq m:\ d(x_{\beta(n)},x_{\beta(m)})<2^{-n-2}$, then there is an α in σ such that for all $n\in\mathbb{N}:\ (\alpha(n))_0=\beta(n)$. This is a straightforward consequence of the definition of σ and the fact that h realizes ($\star$). Now for α,β in σ put $d(\alpha,\beta)\underset{D}{=} d_{\mathbb{R}}\text{-lim}(d(x_{\alpha(n)_0},x_{\beta(n)_0}))_{n\in\mathbb{N}}$. Let a be in $\overline{\overline{\sigma}}$, $lg(a)\geq 1$, and let α, β in $\sigma\cap a$. Then for all $m\in\mathbb{N}$, $m>lg(a)$: $d(x_{a_{lg(a)-1}},x_{\alpha(m)_0})<2^{-lg(a)-2}$ and $d(x_{a_{lg(a)-1}},x_{\beta(m)_0})<2^{-lg(a)-2}$. Therefore $d(\alpha,\beta)\leq 2^{-lg(a)-1}<2^{-lg(a)}$. From these observations it follows that (σ,d) is a steady metric spread, which coincides with $\overline{(X,d)}$ •

3.0.3 remember that we used $\mathbf{AC}_{11}$ to prove that a function from a topological spread $(\rho,\mathcal{T})$ to an arbitrary topological space $(Y,\mathcal{T}')$ can be represented by a spread-function (see 0.1.5). The previous theorem allows us to use only $\mathbf{AC}_{10}$ (and $\mathbf{AC}_{01}$), to prove that a function from a topological spread $(\rho,\mathcal{T})$ to a metric space (X,d) can be represented by a spread-function.

LEMMA: let f be a function from a topological spread $(\rho,\mathcal{T})$ to a metric space (X,d). Then f can be represented by a spread-function γ from $(\rho,\mathcal{T})$ to (σ,d), where (σ,d) is the metric spread constructed in the proof of theorem 3.0.2 coinciding with $\overline{(X,d)}$.

PROOF: let $(x_n)_{n\in\mathbb{N}}$ be as in the proof of theorem 3.0.2. Let $m\in\mathbb{N}$. We have:

($\star$) $\forall\alpha\in\rho\ \exists n\in\mathbb{N}\ [\,d(f(\alpha),x_n)<2^{-m-5}\,]$

By $\mathbf{AC}_{10}$ there is a spread-function γ from ρ to $\mathbb{N}$ realizing ($\star$). Since $m\in\mathbb{N}$ is arbitrary, we find:

($\star\star$) $\forall m\in\mathbb{N}\ \exists\gamma\in\sigma_\omega\ [\,\gamma$ realizes $(\star)\,]$

By $\mathbf{AC}_{01}$ there is a sequence $(\gamma_m)_{m\in\mathbb{N}}$ of spread-functions from ρ to $\mathbb{N}$ such that for all $m\in\mathbb{N}$: γ_m realizes ($\star$) for m. Now define a spread-function γ from $(\rho,\mathcal{T})$ to (σ,d) by putting, for α in ρ and $m\in\mathbb{N}$: $\gamma(\alpha)(m)\underset{D}{=}\langle\!\langle\gamma_m(\alpha),m+4\rangle\!\rangle$. It is straightforward to check that γ is as required •

REMARK: the lemma is a generalization of the last remark in [Kleene&Vesley65, chapt.I,§7.15], which says the following. If A is a subset of $\sigma_\omega\times\sigma_\omega$ such that:

($\star$) $\forall\alpha\in\sigma_\omega\ \exists!\beta\in\sigma_\omega\ [\,(\alpha,\beta)\in A\,]$

Then the existence of a spread-function γ from σ_ω to σ_ω realizing ($\star$) is already a consequence of $\mathbf{AC}_{01}$ combined with $\mathbf{AC}_{10}$. Notice that by definition A is a function from $(\sigma_\omega, d_\omega)$ to $(\sigma_\omega, d_\omega)$.

3.1 OPEN COVERS OF (X, d)

3.1.0* in this section we unfold our most powerful topological tools: star-finite open covers and partitions of unity. These have been left almost undiscussed (at least to our knowledge) in the constructive/intuitionistic literature. Only in [Troelstra66] the existence of a star-finite refinement of an arbitrary open cover of a (1-)locally compact space is proved. We will prove the same for every metric spread, dropping the condition 'locally compact'. A number of our results are not-too-difficult adaptations of classical theorems dating back to the forties and fifties, see for instance [Morita48] and [vanMill89, 3.6.17].

DEFINITION: let U be a subset of (X, d). Then U is *enumerably open* in (X, d) iff there is a sequence $(x_n)_{n\in\mathbb{N}}$ in X, and a sequence $(\rho_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{\geq 0}$, such that $U = \bigcup_{n\in\mathbb{N}} B(x_n, \rho_n)$. Now let $\mathcal{U}$ be an open cover of (X, d) (def. 1.1.4). Then $\mathcal{U}$ is called *per-enumerable* iff $\mathcal{U}$ is an enumerable collection of enumerably open subsets.

LEMMA: let $U = \bigcup_{n\in\mathbb{N}} B(x_n, \rho_n)$ be enumerably open in (X, d). Then there is a continuous function f from (X, d) to $([0,1], d_{\mathbb{R}})$ such that $f^{-1}((0,1]) = U$

PROOF: define f by: $f(x) \underset{D}{=} \frac{1}{2} \sum_n \dfrac{\sup(0, \rho_n - d(x, x_n))}{1 + \sup(0, \rho_n - d(x, x_n))} \cdot 2^{-n}$ for x in X •

3.1.1* DEFINITION: let $\mathcal{U} = \{U_n \mid n\in\mathbb{N}\}$ be an enumerable open cover of (X, d). Then $\mathcal{U}$ is called (i) *locally finite*, (ii) *star-finite*, and (iii) *strongly star-finite* iff

(i) $\forall x\in X\ \exists n, N\in\mathbb{N}\ \forall y\in B(x, 2^{-n})\ \forall m\in\mathbb{N}\ \forall z\in U_m\ [m > N \rightarrow y \# z]$

(ii) $\forall n\in\mathbb{N}\ \exists N\in\mathbb{N}\ \forall m\in\mathbb{N}\ [m > N \rightarrow \forall x\in U_n \forall y\in U_m [x \# y]]$.

(iii) $\forall n\in\mathbb{N}\ \exists N, s\in\mathbb{N}\ \forall m\in\mathbb{N}\ [m > N \rightarrow \forall x\in U_n \forall y\in U_m [d(x, y) > 2^{-s}]]$.

THEOREM: let $\mathcal{U} = \{U_n \mid n\in\mathbb{N}\}$ be a per-enumerable open cover of (X, d). Then there is a strongly star-finite refinement of $\mathcal{U}$.

PROOF: we have:

(⋆) $\forall n\in\mathbb{N}\ \exists(\alpha,\beta)\in X^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}_{\geq 0}\ [U_n=\bigcup_{m\in\mathbb{N}}B(\alpha(m),\beta(m))]$

Using $\mathbf{AC}_{01}$ determine a sequence $(x_{n,m})_{n,m\in\mathbb{N}}$ in X, and a sequence $(\rho_{n,m})_{n,m\in\mathbb{N}}$ in $\mathbb{R}_{\geq 0}$ such that for each $n\in\mathbb{N}$: $U_n=\bigcup_m B(x_{n,m},\rho_{n,m})$. For each $n\in\mathbb{N}$ define a function f_n from (X,d) to $([0,1],d_{\mathbb{R}})$ as in the proof of lemma 3.1.0 such that $f_n^{-1}((0,1]) = U_n$. Define a continuous function f from (X,d) to $([0,1],d_{\mathbb{R}})$ by putting: $f(x) \underset{D}{=} \frac{1}{2}\sum_n f_n(x)\cdot 2^{-n}$. Notice that for all x in X: $0<f(x)<1$ (remember that $\mathcal{U}$ is an open cover).

$\boxed{\text{claim}}$ $\forall x\in X\ \forall k\in\mathbb{N}\ [f(x)>\frac{1}{k+1}\rightarrow\exists j\leq k\,[x\in U_j]]$.

$\boxed{\text{proof}}$ trivial ∘

For $s\in\mathbb{N}_{\geq 2}$ and $t\in\{0,\dots,s\}$ put $V_{s,t}=f^{-1}((\frac{1}{s+1},\frac{1}{s-1}))\cap U_t$. By the claim $\mathcal{V}=\{V_{s,t}\,|\,s\in\mathbb{N}_{\geq 2},\,t\in\{0,\dots,s\}\}$ is an open cover of (X,d), and trivially $\mathcal{V}$ is a refinement of $\mathcal{U}$. Let k be a bijection from $\mathbb{N}$ to $\{(s,t)\,|\,s\in\mathbb{N}_{\geq 2},\,t\in\{0,\dots,s\}\}$. Put $V_n=V_{k(n)}$ for $n\in\mathbb{N}$, then $\mathcal{V}=\{V_n\,|\,n\in\mathbb{N}\}$.

$\boxed{\text{claim}}$ $\mathcal{V}$ is strongly star-finite.

$\boxed{\text{proof}}$ let $n\in\mathbb{N}$, and x in V_n which is equal to $V_{s,t}$ for some $s\in\mathbb{N}_{\geq 2}$ and $t\in\{0,\dots,s\}$. There are but finitely many $m\in\mathbb{N}$ such that $(k(m))_0\in\{s-1,s+1\}$. Therefore we can find an $N\in\mathbb{N}$ such that for all $m\in\mathbb{N}$, $m>N$ and all y in X: $y\in V_m$ implies $f(y)\leq\frac{1}{s+2}\vee f(y)\geq\frac{1}{s-2}$. On the other hand let z be in X such that $d(x,z)<\frac{1}{s+1}-\frac{1}{s+2}$. Then for all $m\in\mathbb{N}$: $d_{\mathbb{R}}(f_m(x),f_m(z))<\frac{1}{s+1}-\frac{1}{s+2}$, by the definition of f_m and the triangle inequality. So $d_{\mathbb{R}}(f(x),f(z))<\frac{1}{s+1}-\frac{1}{s+2}$, meaning that $f(z)\in(\frac{1}{s+2},\frac{1}{s-2})$. Combining these observations we obtain that for all $m\in\mathbb{N}$, $m>N$ and all y in V_m: $d(x,y)\geq\frac{1}{s+1}-\frac{1}{s+2}$ ∘ •

3.1.2* there is a special attraction in per-enumerable covers, in that they possess a subordinate partition of unity, a feature explained in the following definition.

DEFINITION: let $\mathcal{U}$ be an open cover of (X,d), and let $(p_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions from (X,d) to $([0,1],d_{\mathbb{R}})$. We say that $(p_n)_{n\in\mathbb{N}}$ is a *partition of unity* iff $\{p_n^{-1}((0,1])\,|\,n\in\mathbb{N}\}$ is a locally finite open cover of (X,d) and for all x in X: $\sum_n p_n(x)\equiv 1$. In addition $(p_n)_{n\in\mathbb{N}}$ is called *subordinate* to $\mathcal{U}$ iff for all $n\in\mathbb{N}$ there is a U in $\mathcal{U}$ such that: $p_n^{-1}((0,1])\subseteq U$.

LEMMA: let $\mathcal{U}=\{U_n \mid n\in\mathbb{N}\}$ be an enumerable open cover of (X,d), and let $(q_n)_{n\in\mathbb{N}}$ be a partition of unity subordinate to $\mathcal{U}$. Then there is a partition of unity $(p_n)_{n\in\mathbb{N}}$ such that for all $n\in\mathbb{N}$: $p_n^{-1}((0,1])\subseteq U_n$.

PROOF: we have by definition:

$$(\star) \quad \forall m\in\mathbb{N}\ \exists s\in\mathbb{N}\ [\, q_m^{-1}((0,1])\subseteq U_s \,]$$

By $\mathbf{AC}_{00}$ there is a function h from $\mathbb{N}$ to $\mathbb{N}$ realizing $(\star)$. Define, for $n\in\mathbb{N}$, a continuous function p_n from (X,d) to $([0,1],d_{\mathbb{R}})$ by:

$$p_n(x) \underset{D}{=} \textstyle\sum_{m,h(m)=n} q_m(x)\,.$$

Notice that this is a sensible definition since $(q_n)_{n\in\mathbb{N}}$ is a partition of unity. So for any x in X there is an $M\in\mathbb{N}$ such that for all $m\in\mathbb{N}$, $m>M$: $q_m(x)\equiv 0$. And in order to calculate $p_n(x)$ we only need to consider $\{h(m)\mid m\in\mathbb{N},\ m\leq M\}$. This reasoning also shows that $(p_n)_{n\in\mathbb{N}}$ is a partition of unity, and trivially it satisfies our requirement •

THEOREM: let $\mathcal{U}=\{U_n \mid n\in\mathbb{N}\}$ be a per-enumerable open cover of (X,d). Then there is a partition of unity subordinate to $\mathcal{U}$.

PROOF: we copy the notations from the proof of theorem 3.1.1. For each $s\in\mathbb{N}_{\geq 2}$ and $t\in\{0,\ldots,s\}$ we define a continuous function $q_{s,t}$ from (X,d) to $([0,1],d_{\mathbb{R}})$ by putting:

$$q_{s,t}(x) \underset{D}{=} \sup(\inf(\{\tfrac{1}{s-1}-f(x),\ f(x)-\tfrac{1}{s+1},\ f_t(x)\},\ 0)$$

Observe that for all x in X: $q_{s,t}(x)>0$ iff $x\in V_{s,t}$. Now for $n\in\mathbb{N}$ we define a continuous q_n from (X,d) to $([0,1],d_{\mathbb{R}})$ by:

$$q_n(x) \underset{D}{=} \frac{q_{k(n)}(x)}{\sum_m q_{k(m)}(x)}$$

It is straightforward to check that $(q_n)_{n\in\mathbb{N}}$ is as required •

3.1.3 THEOREM: every open cover of a spreadlike metric space has a per-enumerable refinement.

PROOF: it suffices to prove the theorem for a given metric spread (σ,d). Clearly $\{B(\alpha_a,2^{-n}) \mid a\in\overline{\overline{\sigma}},\ n\in\mathbb{N}\}$ is an enumerable basis of (σ,d). By proposition 1.1.6 there are functions h_0 and h_1 from $\mathbb{N}$ to $\overline{\overline{\sigma}}$ and $\mathbb{N}$ respectively such that

$\{B(\alpha_{h_0(n)}, 2^{-h_1(n)}) \mid n\in\mathbb{N}\}$ is a refinement of $\mathcal{U}$. Trivially this refinement is per-enumerable •

COROLLARY: let $\mathcal{U}$ be an open cover of a spreadlike metric space, then:

(i) there is a strongly star-finite refinement of $\mathcal{U}$, using theorem 3.1.1.

(ii) there is a partition of unity subordinate to $\mathcal{U}$, using theorem 3.1.2.

3.1.4* in Bishop's school the following lemma can sometimes serve as a substitute of the previous theorem. We use it in the proof of lemma 4.1.0 which is a prelude to the Dugundji theorem (4.1.1).

LEMMA: let $\mathcal{U}$ be an open cover of (X,d). Let $(a_n)_{n\in\mathbb{N}}$ be dense in (X,d), and suppose: $(\rho_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^+$ and $(V_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathcal{U}$ such that

(i) $B(a_n,\rho_n) \subseteq V_n$.

(ii) if h is a function from $\mathbb{N}$ to $\mathbb{N}$ such that $(a_{h(n)})_{n\in\mathbb{N}}$ is d-Cauchy then there is $\delta\in\mathbb{R}^+$ such that $\forall n\in\mathbb{N}\,[\,\rho_{h(n)}>\delta\,]$

then $\mathcal{V}=(B(a_n,\rho_n))_{n\in\mathbb{N}}$ is a per-enumerable refinement of $\mathcal{U}$.

PROOF: the only nontrivial concern is that $\mathcal{V}$ be an open cover, so we show: for all x in X there is an $m\in\mathbb{N}$ such that x is in $B(a_m,\rho_m)$. Since $(a_n)_{n\in\mathbb{N}}$ is dense in (X,d), we have:

($\star$) $\forall n\in\mathbb{N}\,\exists m\in\mathbb{N}\,[\,d(x,a_m)<2^{-n}\,]$

By $\mathbf{AC}_{00}$ there is a function h from $\mathbb{N}$ to $\mathbb{N}$ realizing ($\star$). Then $(a_{h(n)})_{n\in\mathbb{N}}$ is d-Cauchy so there is $\delta\in\mathbb{R}^+$ such that $\forall n\in\mathbb{N}\,[\,\rho_{h(n)}>\delta\,]$, by (ii). Determine $n\in\mathbb{N}$ such that $d(x,a_{h(n)})<\delta$. Then x is in $B(a_{h(n)},\rho_{h(n)})$ •

3.1.5 our first application of the previous theorems concerns the normality of a metric spread. Recall that a topological space $(X,\mathcal{T})$ is normal iff for all U, V in $\mathcal{T}$: if $U\cup V=X$ then there are W, Z in $\mathcal{T}$ such that $U\cup W=X=V\cup Z$ and $W\cap Z=\emptyset$.

THEOREM: every spreadlike metric space is normal.

PROOF: it suffices to prove that a given metric spread (σ,d) is normal. Let U, V be open in (σ,d) such that $U\cup V=\sigma$. We have:

(⋆) $\forall\alpha\in\sigma\ \exists(s,n)\in\{0,1\}\times\mathbb{N}\ [\,(s=0\wedge B(\alpha,2^{-n+1})\subset U)\vee(s=1\wedge B(\alpha,2^{-n+1})\subset V)\,]$

By $\mathbf{AC}_{10}$ there is a spread-function γ from σ to $\{0,1\}\times\mathbb{N}$ realizing (⋆). Let γ_0 and γ_1 be spread-functions from σ to $\{0,1\}$ and $\mathbb{N}$ respectively such that for all α in σ: $\gamma(\alpha)=(\gamma_0(\alpha),\gamma_1(\alpha))$. Then $\mathcal{U}=\{B(\alpha,2^{-\gamma_1(\alpha)})\mid\alpha\in\sigma\}$ is an open cover of (σ,d), so by corollary 3.1.3 there is a strongly star-finite refinement $\mathcal{W}=\{W_n\,|\,n\in\mathbb{N}\}$ of $\mathcal{U}$. The construction of $\mathcal{W}$ gives a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in σ such that for each $n\in\mathbb{N}$: $W_n\subseteq B(\alpha_n,2^{-\gamma_1(\alpha_n)})$.

Let α be in σ. Determine $n\in\mathbb{N}$ such that $\alpha\in W_n$. Determine $s,N\in\mathbb{N}$ such that for all $m\in\mathbb{N}$: $m>N$ implies $\forall\beta\in W_n\ \forall\gamma\in W_m\ [\,d(\beta,\gamma)>2^{-s}\,]$. We can decide:

$\boxed{\text{case 1}}$ $\exists m\leq N\ [\,\gamma_0(\alpha)\neq\gamma_0(\alpha_m)\wedge d(\alpha,\alpha_m)<2^{-\gamma_1(\alpha_m)+1}\,]$.
Then we find that α is in $U\cap V$, by our construction of γ.

$\boxed{\text{case 2}}$ $\forall m\leq N\ [\,\gamma_0(\alpha)\neq\gamma_0(\alpha_m)\rightarrow d(\alpha,\alpha_m)>2^{-\gamma_1(\alpha_m)})\,]$.
Then $\epsilon=\inf(\{d(\alpha,\alpha_m)-2^{-\gamma_1(\alpha_m)}\mid m\leq N,\ \gamma_0(\alpha)\neq\gamma_0(\alpha_m)\})$ is in $\mathbb{R}^+$. So we can find $t\in\mathbb{N}$ such that $B(\alpha,2^{-t})\subseteq W_n$ and moreover $2^{-t}\leq\inf(2^{-s},\epsilon)$. This means that $B(\alpha,2^{-t})\cap W_m=\emptyset$ for all $m\in\mathbb{N}$ for which $\gamma_0(\alpha)\neq\gamma_0(\alpha_m)$, and a fortiori $\gamma_0(\alpha)=\gamma_0(\alpha_n)$.

Put together this gives us:

(⋆⋆) $\forall\alpha\in\sigma\ \exists(s,t,n)\in\{0,1\}\times\mathbb{N}\times\mathbb{N}\ [\,(s=0\wedge\alpha\in U\cap V)\ \vee\ (s=1\wedge B(\alpha,2^{-t})\subseteq W_n\ \wedge$
$\forall m\in\mathbb{N}\,[\gamma_0(\alpha)\neq\gamma_0(\alpha_m)\rightarrow B(\alpha,2^{-t})\cap W_m=\emptyset])\,]$

By $\mathbf{AC}_{10}$ there is a spread-function δ from σ to $\{0,1\}\times\mathbb{N}\times\mathbb{N}$ realizing (⋆⋆). Let δ_0, δ_1 and δ_2 be spread-functions from σ to $\{0,1\}$, $\mathbb{N}$ and $\mathbb{N}$ respectively such that for all α in σ: $\delta(\alpha)=(\delta_0(\alpha),\delta_1(\alpha),\delta_2(\alpha))$.

Put $A=\{\alpha\in\sigma\,|\,\gamma_0(\alpha)=0\wedge\delta_0(\alpha)=1\}$ and $C=\{\alpha\in\sigma\,|\,\gamma_0(\alpha)=1\wedge\delta_0(\alpha)=1\}$. Let $W=\bigcup_{\alpha\in C}B(\alpha,2^{-\delta_1(\alpha)})$ and $Z=\bigcup_{\alpha\in A}B(\alpha,2^{-\delta_1(\alpha)})$. Clearly W and Z are open in (σ,d). We show that $\sigma=U\cup W$. Let α be in σ. If $\gamma_0(\alpha)=0$ or $\delta_0(\alpha)=0$ then α is in U. But else $\gamma_0(\alpha)=1$ and $\delta_0(\alpha)=1$, meaning that α is in C and so in W. Similarly $\sigma=V\cup Z$. It remains to verify that $W\cap Z=\emptyset$. For this let η' and ζ' be arbitrary elements of W and Z respectively. Determine $\eta\in C$ and $\zeta\in A$ such that $\eta'\in B(\eta,2^{-\delta_1(\eta)})$ and $\zeta'\in B(\zeta,2^{-\delta_1(\zeta)})$. Then $B(\eta,2^{-\delta_1(\eta)})\subseteq W_{\delta_2(\eta)}$ and $\gamma_0(\alpha_{\delta_2(\eta)})=\gamma_0(\eta)\neq\gamma_0(\zeta)$. Therefore $B(\zeta,2^{-\delta_1(\zeta)})\cap W_{\delta_2(\eta)}=\emptyset$, meaning $\zeta'\#\eta'$. This shows that $W\cap Z=\emptyset$ •

3.1.6 we wish to prove that every complete metric space (X,d) coincides with a weakly star-finite apartness spread $(\rho,\#)$. The idea behind this theorem is not so difficult. First we must know that for a complete metric space (X,d) the metric topology refines the apartness topology $\mathcal{T}_{\#_d}$ (and of course vice versa). In other words: (X,d) coincides identically with $(X,\#_d)$. This is theorem and corollary 3.3.10. We start out with a steady metric spread (σ,d) coinciding isometrically with (X,d). Then we have that $(\alpha_a)_{a\in\overline{\overline{\sigma}}}$ is dense in (σ,d), so there is a canonical sequence $(a_n)_{n\in\mathbb{N}}$ which is dense in (σ,d). We construct a strongly star-finite refinement of $\{B(a_n,2^{-0})\,|\,n\in\mathbb{N}\}$. Then we construct a strongly star-finite refinement of $\{B(a_n,2^{-1})\,|\,n\in\mathbb{N}\}$, and then of $\{B(a_n,2^{-2})\,|\,n\in\mathbb{N}\}$, etcetera. We obtain a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ of strongly star-finite covers such that for each V in $\mathcal{V}_n$: $diam(V)<2^{-n}$. Now we use the elements of $\mathcal{V}_n$ as the n^{th} nodes in our spread ρ. It is straightforward to define a weakly star-finite touch-relation $\approx$ on $\overline{\overline{\rho}}$ such that for the corresponding apartness $\#$ we have: (σ,d) coincides with $(\rho,\#)$.

Our proof, in order to be precise and correct, involves some more work than one might expect. To make our usage of the axioms in the proof impeccable we first study our theorem 3.1.3 in close detail (proving more than necessary).

LEMMA: let $\mathcal{U}$ be an open cover of (σ,d). Let $(a_n)_{n\in\mathbb{N}}$ be dense in (σ,d), and let for $n\in\mathbb{N}$: $b_n=a_{(n)_0}$. Then there is a β in σ_ω such that $B(b_n,2^{-\beta(n)})$ is a refinement of $\mathcal{U}$.

PROOF: let γ in σ_ω be the spread-function constructed in the proof of theorem 3.1.3. We have:

$(\star)$ $\quad \forall n\in\mathbb{N}\;\exists m\in\mathbb{N}\;\exists U\in\mathcal{U}\;[\,B(b_n,2^{-m})\subseteq U\,]$

By $\mathbf{AC}_{00}$ there is an h in σ_ω realizing $(\star)$. Now define β in σ_ω as follows. Let $n\in\mathbb{N}$, then:

$$\beta(n) \underset{D}{=} \min(\{h(n)\}\cup\{(\gamma(i)-1)_1\,|\,i\leq n,\ i\in\overline{\overline{\sigma}},\ (\gamma(i)-1)_0=(n)_0\}).$$

It is straightforward to check that β is as required $\bullet$

3.1.7 next, observe that if $(a_n)_{n\in\mathbb{N}}$ is dense in (σ,d), and β is in σ_ω such that $\{B(a_n,2^{-\beta(n)})\,|\,n\in\mathbb{N}\}$ is an open cover of (σ,d), then we can canonically construct a strongly star-finite refinement $\mathcal{V}$ of $\{B(a_n,2^{-\beta(n)})\,|\,n\in\mathbb{N}\}$. For we can put $x_{n,m}=a_n$ and $\rho_{n,m}=\beta(n)$, and then follow the instructions as put forward in the proof of theorem 3.1.1. So in fact $\mathcal{V}=\mathcal{V}^\beta$ is nothing but the set $P=\{(s,t)\in\mathbb{N}\times\mathbb{N}\,|\,s\geq 2,\ t\leq s\}$ along with a spread-function γ assigning to each α in σ an $(s,t)\in P$ such that α is in $V^{\beta}_{(s,t)}$. For we have:

$(\star)$ $\forall\alpha\in\sigma\;\exists(s,t)\in P\;[\,\alpha\in V^{\beta}_{(s,t)}\,]$

so by $\mathbf{AC}_{10}$ there is a spread-function γ in σ_ω realizing $(\star)$. We obtain:

LEMMA: let $(a_n)_{n\in\mathbb{N}}$ be dense in (σ,d), and let $\{B(a_n,2^{-\beta(n)})\,|\,n\in\mathbb{N}\}$ be an open cover of (σ,d). Then there is a spread-function γ from σ to P such that for all α in σ: $\alpha\in V^{\beta}_{\gamma(\alpha)}$.

PROOF: this is just our discussion above •

COROLLARY: let $\mathcal{U}$ be an open cover of (σ,d). Let $(a_n)_{n\in\mathbb{N}}$ be dense in (σ,d), and let for $n\in\mathbb{N}$: $b_n=a_{(n)_0}$. Then there are β and γ in σ_ω such that

(i) $\{B(b_n,2^{-\beta(n)})\,|\,n\in\mathbb{N}\}$ is a refinement of $\mathcal{U}$.

(ii) γ is a spread-function from σ to P such that for all α in σ: $\alpha\in V^{\beta}_{\gamma(\alpha)}$.

PROOF: combine the above lemma with lemma 3.1.6 •

REMARK: we will not use the full strength of this corollary, but observe that it shows how to reduce an open cover $\mathcal{U}$ of (σ,d) to an element of σ_ω which encodes a strongly star-finite refinement of $\mathcal{U}$. This means we can apply $\mathbf{AC}_{01}$, $\mathbf{AC}_{11}$, and $\mathbf{DC}_1$ in appropriate situations, such as the next theorem. Other situations are e.g. if we know that for each α in σ there is an open cover $\mathcal{U}$ with special properties with respect to α, or e.g. if we wish to construct consecutive refinements $(\mathcal{V}^n)_{n\in\mathbb{N}}$ of an open cover $\mathcal{U}$, where $\mathcal{V}^{n+1}$ depends essentially on the choice of $\mathcal{V}^n$.

3.1.8 THEOREM: let (X,d) be a complete metric space. Then (X,d) coincides with a weakly star-finite apartness spread $(\rho,\#)$.

PROOF: by theorem 3.0.2 (X,d) coincides with a metric spread (σ,d). Let $(a_n)_{n\in\mathbb{N}}$ be dense in (σ,d). For $n\in\mathbb{N}$ put $\mathcal{U}_n=\{B(a_m,2^{-n-2})\,|\,m\in\mathbb{N}\}$, and let $\beta_n=\underline{n}$. Simply write $\mathcal{V}^n=\{V^n_{(s,t)}\,|\,(s,t)\in P\}$ for the canonical star-finite refinement $\mathcal{V}^{\beta_n}$ of $\mathcal{U}_n$ as described in 3.1.7. By lemma 3.1.7 we obtain:

$(\star)$ $\forall n\in\mathbb{N}\;\exists\gamma\in\sigma_\omega\;[\,\gamma$ is a spread-function from σ to $P\wedge\forall\alpha\in\sigma\,[\alpha\in V^n_{\gamma(\alpha)}]\,]$

Then by $\mathbf{AC}_{01}$ we find a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in σ_ω realizing $(\star)$. To construct our promised spread ρ we wish to allow as n^{th} nodes precisely $\{\gamma_n(\alpha)\,|\,\alpha\in\sigma\}$ which equals $\{\gamma_n(m)-1\,|\,m\in\mathbb{N},\ \gamma_n(m)>0\}$. But of course these n^{th} nodes belong to particular $(n-1)^{\text{th}}$ nodes only, so we must be precise. Therefore we define a spread ρ as follows. Let a in $\mathbb{N}$, then:

$$\rho(a) \underset{D}{=} \begin{cases} 0 & \text{if} \quad \forall i<lg(a)\ [\,(a_i)_1=\mu s\in\overline{\overline{\sigma}}\,[\forall j\leq i\,\exists k\sqsubseteq s\,[\gamma_j(k)=(a_j)_0+1]]\,] \\ 1 & \text{else} \end{cases}$$

Notice that $\rho(\langle\!\langle\rangle\!\rangle)=0$. Let $a\in\mathbb{N}$ such that $\rho(a)=0$, then $\sigma((a_{lg(a)-1})_1)=0$. Put $b=(a_{lg(a)-1})_1$. Determine $p\in\mathbb{N}$ such that $p=\gamma_{lg(a)}(\alpha_b)$, and also determine $s=\mu t\in\overline{\overline{\sigma}}\,[\forall j\leq lg(a)-1\,\exists k\sqsubseteq s\,[\gamma_j(k)=(a_j)_0+1]\,]\wedge\gamma_{lg(a)}(s)=p+1]\,]$. Then by definition $\rho(a\star\langle\!\langle p,s\rangle\!\rangle)=0$. So we see that ρ is indeed a spread. Let α be in ρ, then $((\alpha(n))_0)_{n\in\mathbb{N}}$ codes a sequence of inhabited open sets $(V^n_{s_n,t_n})_{n\in\mathbb{N}}$ such that for each $n\in\mathbb{N}$: $d(\beta,\gamma)<2^{-n-1}$ for all β, γ in $V^n_{s_n,t_n}$. Put $b_n=(\alpha(n))_1$ for $n\in\mathbb{N}$. Then for all $i\leq n$ we have that α_{b_n} is in $V^i_{s_i,t_i}$. This shows that for all $m\in\mathbb{N}$: $d(\alpha_{b_n},\alpha_{b_{n+m}})<2^{-n-1}$. Define a function j from (ρ, d_ω) to (σ, d) by defining $j(\alpha)$ as follows:

$$j(\alpha) \underset{D}{=} d\text{-}\lim(\alpha_{b_n})_{n\in\mathbb{N}}$$

Then $j(\alpha)\in\sigma$, since (σ,d) is complete, and for all $n\in\mathbb{N}$: $d(j(\alpha),\alpha_{b_n})\leq 2^{-n-1}$. Moreover j is surjective. For let β be in σ. From our definitions it is clear that there is an α in ρ such that for all $n\in\mathbb{N}$: $(\alpha(n))_0=\gamma_n(\beta)$. Then $j(\alpha)\equiv\beta$ since for all $n\in\mathbb{N}$: $d(\alpha_{b_n},\beta)<2^{-n-1}$, where $b_n=(\alpha(n))_1$. So, defining d on ρ by putting, for α,β in ρ: $d(\alpha,\beta)=d(j(\alpha),j(\beta))$, we have that (σ,d) coincides with (ρ,d).

It remains for us to define a weakly star-finite touch-relation $\approx$ on $\overline{\overline{\rho}}$ which induces the apartness $\#_d$ on ρ. To this end we first define a binary symmetric relation $\approxeq$ on P as follows. Let (s,t), $(p,r)\in P$, then:

$(s,t)\approxeq(p,r)$ iff $p\in\{s-1,\ s,\ s+1\}$.

Next let $n\in\mathbb{N}$. We have:

$(\star)$ $\quad\forall a,b\in\overline{\overline{\sigma}}\ \exists s\in\mathbb{N}\ [\,(s=0\wedge d(\alpha_a,\alpha_b)<2^{-n+1})\vee(s=1\wedge d(\alpha_a,\alpha_b)>2^{-n})\,]$

By $\mathbf{AC}_{00}$ there is a function h from $\overline{\overline{\sigma}}\times\overline{\overline{\sigma}}$ to $\{0,1\}$ realizing $(\star)$. So we find:

$(\star\star)$ $\quad\forall n\in\mathbb{N}\ \exists h\in\sigma_\omega\ [\,h$ realizes $(\star)\,]$

By $\mathbf{AC}_{01}$ there is a γ in σ_ω realizing ($\star\star$), meaning there is a sequence $(h_n)_{n\in\mathbb{N}}$ of functions from $\overline{\overline{\sigma}}\times\overline{\overline{\sigma}}$ to $\{0,1\}$ such that for each $n\in\mathbb{N}$: h_n realizes ($\star$). Good! We hope you still have energy left for the final step: defining $\approx$. For all a in $\overline{\overline{\rho}}$ put $a\approx\lessdot\gtrdot\approx a$. Now let a, b be in $\overline{\overline{\rho}}$ such that $0<lg(a)\leq lg(b)$. Let $s=lg(a)$, $t=lg(b)$ and define:

$$a\approx b \text{ iff } b\approx a \text{ iff } h_s((a_{s-1})_1,(b_{s-1})_1)=0 \text{ and for all } i<s:\ (a_i)_0\approxeq(b_i)_0\,.$$

Clearly $\approx$ is a decidable symmetric subset of $\overline{\overline{\rho}}\times\overline{\overline{\rho}}$ such that $\not\approx$ is monotone. So $\approx$ satisfies definition 2.0.2 (i). To see that $\approx$ satisfies definition 2.0.2 (ii), let α, β be in ρ. Suppose s, $t\in\mathbb{N}$ are such that $a=\overline{\alpha}(s)\not\approx\overline{\beta}(t)=b$. Without loss of generality $s\leq t$.

$\boxed{\text{case 1}}$ $h_s((a_{s-1})_1,(b_{s-1})_1)=1$.
Then $d(\alpha_{(a_{s-1})_1},\alpha_{(b_{s-1})_1})>2^{-s}$. But $d(j(\alpha),\alpha_{(a_{s-1})_1})\leq 2^{-s-1}$ and $d(j(\beta),\alpha_{(b_{s-1})_1})\leq 2^{-s-1}$, by our earlier remark. Then $d(j(\alpha),j(\beta))>0$, so $d(\alpha,\beta)>0$.

$\boxed{\text{case 2}}$ there is $i<s$ such that $(a_i)_0\not\approxeq(b_i)_0$.
Then it follows from the proof of theorem 3.1.1 that there is $m\in\mathbb{N}$ such that for all $\alpha'\in V^i_{(a_i)_0}$ and all $\beta'\in V^i_{(b_i)_0)}$: $d(\alpha',\beta')>2^{-m}$. But for all $n\in\mathbb{N}$, $n\geq i$ we have that $\alpha_{(\overline{\alpha}(n))_1}\in V^i_{(a_i)_0}$ and $\alpha_{(\overline{\beta}(n))_1}\in V^i_{(b_i)_0}$. So we see $d(j(\alpha),j(\beta))>0$, therefore $d(\alpha,\beta)>0$.

On the other hand, if $d(\alpha,\beta)>0$, then it is easy to see that there is $n\in\mathbb{N}$ such that $\overline{\alpha}(n)\not\approx\overline{\beta}(n)$. We obtain, for α, β in ρ: $\alpha\#_d\beta$ iff $\exists n\in\mathbb{N}\ [\overline{\alpha}(n)\not\approx\overline{\beta}(n)]$. This ensures that $\approx$ satisfies definition 2.0.2 (ii), and so is a touch-relation on ρ, which induces the apartness $\#_d$.

$\boxed{\text{claim}}$ $\approx$ is weakly star-finite.

$\boxed{\text{proof}}$ for all $(s,t)\in P$ the set $\{(p,r)\in P\,|\,(s,t)\approxeq(p,r)\}$ contains precisely N_s elements, where $N_s=7$ when $s=2$ and $N_s=3s+2$ else. Also our construction of ρ is such that for each a in $\overline{\overline{\rho}}$: $\{b\in\overline{\overline{\rho}}(lg(a))\,|\,\forall i<lg(a)\ [(b_i)_0=(a_i)_0]\}$ contains only a. This means that for an arbitrary a in $\overline{\overline{\rho}}$ there are at most $\Pi_{i<lg(a)}N_{(a_i)_0}$ elements b of $\overline{\overline{\rho}}(lg(a))$ such that $a\approx b$ $\circ$

Therefore $(\rho,\#_d)$ is weakly star-finite. But by $\mathbf{CP}_{\mathrm{cm}}$ (theorem 3.3.10) d metrizes $(\rho,\#_d)$ and so (σ,d) coincides with $(\rho,\#_d)$ $\bullet$

3.1.9 THEOREM: NOT every complete metric space coincides with a star-finite $(\rho,\#)$.

PROOF: for α in $\sigma_{2\text{mon}}$ put:

$$(X_\alpha,d) = \overline{(\{0\}\cup\{x\in[0,1]\,|\,\exists n\in\mathbb{N}\,[\alpha(n)=1]\},d_{\mathbb{R}})}\,.$$

Then for each α in $\sigma_{2\text{mon}}$, (X_α,d) is a complete metric space. Now let α in $\sigma_{2\text{mon}}$, and suppose (X_α,d) coincides with a star-finite spread $(\rho,\#)$, with corresponding star-finite touch-relation $\approx$ on $\overline{\overline{\rho}}$. Let i be a homeomorphism from (X_α,d) to $(\rho,\#)$. Define d on ρ by putting, for γ,δ in ρ: $d(\gamma,\beta)=d(i^{-1}(\gamma),i^{-1}(\delta))$. Then (ρ,d) is a complete metric spread coinciding with $(\rho,\#)$. Let $\beta=i(0)$. Now consider the subfan $\tau_{\beta,\approx}$ of ρ as defined in 2.4.1. We have:

$$\forall\gamma\in\rho\ \exists s\in\{0,1\}\ [\,(s=0\wedge d(\gamma,\beta)<2^{-1})\vee(s=1\wedge d(\gamma,\beta)>2^{-2})\,]\,.$$

So by **CP** we find:

$$(\star)\qquad \forall\gamma\in\rho\ \exists n\in\mathbb{N}\ \exists s\in\{0,1\}\ \forall\delta\in\rho\ [\,\overline{\delta}(n)=\overline{\gamma}(n)\ \rightarrow\ ((s=0\wedge d(\delta,\beta)<2^{-1})\vee (s=1\wedge d(\delta,\beta)>2^{-2}))\,]$$

In particular $(\star)$ holds for all γ in $\tau_{\beta,\approx}$. So we can apply the fan theorem **FT** to obtain $N\in\mathbb{N}$ such that:

$$(\star\star)\qquad \forall a\in\overline{\overline{\tau}}_{\beta,\approx}(N)\ \exists s\in\{0,1\}\ [\,(s=0\wedge\forall\gamma\in\rho\cap a\,[d(\gamma,\beta)<2^{-1}])\vee (s=1\wedge\forall\gamma\in\rho\cap a\,[d(\gamma,\beta)>2^{-2}])\,]$$

There is a function h from $\overline{\overline{\tau}}_{\beta,\approx}(N)$ to $\{0,1\}$ realizing $(\star\star)$, since $\overline{\overline{\tau}}_{\beta,\approx}(N)$ is finite. Put $A=\{a\in\overline{\overline{\tau}}_{\beta,\approx}(N)\,|\,h(a)=0\wedge a\approx\overline{\beta}(N)\}$. Since $(\rho,\#)$ is star-finite and A is finite, we can determine $n\in\mathbb{N}$ such that $n=\#(\{b\in\overline{\overline{\rho}}\,|\,\exists a\in A\,[b\approx a]\})$.

$\boxed{\text{case 1}}$ $\quad n>\#(A)$.
Then there is a γ in ρ such that $\gamma\#\beta$. So $\alpha\#\underline{0}$.

$\boxed{\text{case 2}}$ $\quad n=\#(A)$.
Then $\rho_A=\{\gamma\in\rho\,|\,\overline{\gamma}(N)\in A\}$ is a decidable inhabited subset of (ρ,d). On the other hand: $\rho_A\subseteq B(\beta,2^{-1})$. Now suppose $\alpha\#\underline{0}$. Then we have that $(X_\alpha,d)\cong([0,1],d_{\mathbb{R}})\cong(\rho,d)$ isometrically, so we find a decidable inhabited subset $i^{-1}(\rho_A)$ of $([0,1],d_{\mathbb{R}})$ with $diam(i^{-1}(\rho_A))<2^{-1}$. Contradiction, see 0.2.2. Therefore $\alpha\equiv\underline{0}$.

Now suppose: for all α in $\sigma_{2\text{mon}}$: (X_α,d) coincides with a star-finite $(\rho,\#)$. Since in our previous discussion α in $\sigma_{2\text{mon}}$ was arbitrary, we then find: $\forall\alpha\in\sigma_{2\text{mon}}\ [\,\alpha\equiv\underline{0}\vee\alpha\#\underline{0}\,]$. Contradiction with **CP**, see 0.0.11 •

3.2 VARIOUS CONCEPTS OF LOCATEDNESS

3.2.0* in this section we investigate various concepts of locatedness. Traditionally a subset (A,d) is located in (X,d) when for each x in X we can compute $\inf(\{d(x,a)\,|\,a\in A\})$. This important concept was introduced by Brouwer. We will show however that 'located in', as might be expected, is not a topological notion. Also, 'located in' does not behave transitively. This leads us to consider four variants of the notion 'located in'. We already introduced 'sublocated in' in chapter one.

We also introduce particular strengthenings of these four variants, similar to the strengthening 'strongly sublocated in' (of 'sublocated in') introduced in chapter one. We end up with the following concepts, in order of strength: best approximable, (strongly) located, (strongly) halflocated, (strongly) sublocated, (strongly) traceable.

'(Strongly) sublocated in' is the first topological notion in the list. We already discussed it in chapter one. The definition of '(strongly) traceable' serves to illustrate the relation between the apartness topology $\mathcal{T}_{\#_d}$ and the metric topology $\mathcal{T}_d$ on X. The most useful concept in our eyes is '(strongly) halflocated in'. This notion behaves transitively, and has a very nice connection with 'strongly sublocated in' (see theorems 4.5.2 and 4.5.3). (Strongly) halflocated subsets crop up very naturally in the course of our investigations in chapter four, where they are seen to be as easily manageable as (strongly) located subsets. We know of no alternative to 'halflocated in' for our results in chapter four. On the other hand, for a strongly compact space (X,d) '(strongly) traceable in (X,d)' implies '(strongly) located in (X,d)', so our definitions are of value mostly in a context of non-strongly-compact spaces.

This section we use to prove some simple necessities and to clarify the definitions by giving examples and counterexamples. Most important in our eyes are lemmas 3.2.2 and 3.2.5, along with proposition 3.2.8 and theorem 3.2.9. We give a definition of 'located in' which is easily seen to be equivalent to the traditional definition, but which opens the door for our adaptions:

DEFINITION: let (A,d) be a subspace of (X,d), a metric space. Then (A,d) is (i) *located*, (ii) *halflocated*, (iii) *sublocated*, (iv) *traceable* in (X,d) iff: (A,d) is inhabited and

(i) $\forall D\in\mathbb{R}_{>1}\ \forall x\in X\ \forall m\in\mathbb{Z}\ [\,\exists a\in A\,[d(x,a)<D^{m+1}]\vee\forall a\in A\,[d(x,a)>D^m]\,]$.

(ii) $\exists D\in\mathbb{R}_{>1}\ \forall x\in X\ \forall m\in\mathbb{Z}\ [\,\exists a\in A\,[d(x,a)<D^{m+1}]\vee\forall a\in A\,[d(x,a)>D^m]\,]$.

(iii) $\forall x\in X\ \forall m\in\mathbb{Z}\ [\,\exists a\in A\,[d(x,a)<2^{m+1}]\vee\exists n\in\mathbb{N}\ \forall a\in A\,[d(x,a)>2^{-n}]\,]$.

(iv) $\forall x \in X\ \forall m \in \mathbb{Z}\ [\exists a \in A\,[d(x,a) < 2^{m+1}] \vee \forall a \in A\,[x \# a]\,]$.

In addition, if $D \in \mathbb{R}_{>1}$ realizes (ii), then we say that (A,d) is halflocated in (X,d) *with parameter* D.

Clearly 'located in' implies 'halflocated in' implies etc. Notice that (iv) is slightly weaker than: '$(A, \#)$ is sublocated in $(X, \#)$' (definition 1.3.3). We will give examples from 3.2.1 onwards, but first we wish to strengthen our definition above in the following way:

3.2.1* DEFINITION: (A,d) is (i) *strongly located*, (ii) *strongly halflocated*, (iii) *strongly sublocated*, (iv) *strongly traceable* in (X,d) iff:

(i) $\forall D \in \mathbb{R}_{>1}\ \forall x \in X\ \exists y \in A\ \forall a \in A\ [\,d(x,y) \leq D \cdot d(x,a)\,]$

(ii) $\exists D \in \mathbb{R}_{\geq 1}\ \forall x \in X\ \exists y \in A\ \forall a \in A\ [\,d(x,y) \leq D \cdot d(x,a)\,]$

(iii) $\forall x \in X\ \exists y \in A\ [\,x \# y\ \rightarrow\ \exists n \in \mathbb{N}\ \forall a \in A\,[d(x,a) > 2^{-n}]\,]$

(iv) $\forall x \in X\ \exists y \in A\ [\,x \# y\ \rightarrow\ \forall a \in A\,[x \# a]\,]$.

In addition, if $D \in \mathbb{R}_{>1}$ realizes (ii), then we say that (A,d) is strongly halflocated in (X,d) *with parameter* D. Finally, if (ii) is realized by $D = 1$, then we say that (A,d) is *best approximable* in (X,d).

REMARK: by lemma 1.3.3 (iv) is the same as '$(A, \#)$ is strongly sublocated in $(X, \#)$' (definition 1.3.3). But of course the apartness topology need not be the same as the metric topology, therefore in general (iv) is different from (iii), which is just a repetition of definition 1.3.3 for the metric topology. Notice that the terminology 'strongly' for (i)-(iv) is justified. For if for x in X, y realizes (ii) with parameter D, then we can always decide: $d(x,y) < D^{m+1}$ or $d(x,y) > D^m$, for $m \in \mathbb{Z}$. But $d(x,y) > D^m$ implies that for all $a \in A$: $d(x,a) > D^{m-1}$. This shows that if (A,d) is strongly halflocated in (X,d), with parameter D, then (A,d) is halflocated in (X,d) with parameter D^2. Then of course 'strongly located in' implies 'located in' since $\{D^2 \mid D \in \mathbb{R}_{>1}\} = \mathbb{R}_{>1}$. The other implications can be obtained in a similar but easier fashion, see 1.3.3. Also notice that if (A,d) is strongly (half,sub)located in (X,d), then (A,d) is closed in (X,d).

The reader probably will benefit from a few examples. They will simultaneously furnish Brouwerian counterexamples to many conjectures which come up naturally in connection with our definitions. In particular we will show that it is daring to say that (iv) implies (iii), etcetera, for this definition as well as for definition 3.2.0. Using **CP** these implications are easily seen to lead to contradiction.

We hope that the examples given in this section are illustrative. But the examples which in our eyes most justify these new definitions will have to wait until chapter four, see 4.2.4.

EXAMPLE: we give a counterexample to the statement 'if (A,d) is strongly traceable in (X,d), then (A,d) is strongly sublocated in (X,d)'. Let $A=\{2^{-n}|n\in\mathbb{N}\}$, let $X=\{0\}\cup A$, and let $d=d_{\mathbb{R}}$.

EXAMPLE: we give a Brouwerian counterexample to the statement 'if (A,d) is strongly traceable in (X,d), then (A,d) is sublocated in (X,d)'. Let $A=\{1\}\cup\{2^{-m}|m\in\mathbb{N}\wedge\exists n\in\mathbb{N}\,[n=k_{99}]\}$ let $X=\{0\}\cup A$, and let $d=d_{\mathbb{R}}$.

3.2.2* LEMMA: (A,d) is (half)located in (X,d) iff $\overline{(A,d)}$ is strongly (half)located in $\overline{(X,d)}$.

PROOF: let (A,d) be halflocated in (X,d), with parameter D in $\mathbb{R}_{>1}$. First we show that (A,d) is halflocated in $\overline{(X,d)}$ with parameter D^3. Let $x=d\text{-}\lim(x_n)_{n\in\mathbb{N}}$ be in $\overline{X}$, where for all $n\in\mathbb{N}$: $d(x,x_n)<D^{-3n+1}-D^{-3n}$. Now let $m\in\mathbb{Z}$. We must show that we can decide: $\exists a\in A\,[d(x,a)<D^{3m+3}]$ or $\forall a\in A\,[d(x,a)>D^{3m}]$. Let $n=|m|$. Since (A,d) is halflocated in (X,d) we can decide:

$\boxed{\text{case 1}}$ there is $a\in A$ such that $d(x_n,a)<D^{3m+2}$.
Then $d(x,a)\leq d(x_n,a)+d(x,x_n)<D^{3m+2}+(D^{3m+3}-D^{3m+2})=D^{3m+3}$.

$\boxed{\text{case 2}}$ for all $a\in A$: $d(x_n,a)>D^{3m+1}$.
Then for all $a\in A$: $d(x,a)\geq d(x_n,a)-d(x,x_n)>D^{3m+1}-(D^{3m+1}-D^{3m})=D^{3m}$.

Next we show that $\overline{(A,d)}$ is strongly halflocated in $\overline{(X,d)}$, with parameter D^6. Let x be in $\overline{X}$. Since (A,d) is halflocated in $\overline{(X,d)}$ with parameter D^3, we can decide:

$\boxed{\text{case 1}}$ for all $a\in A$: $d(x,a)>1$.
Then we can find $s\in\mathbb{N}$ and $y\in A$ such that $d(x,y)<D^{3s+3}$ whereas for all $a\in A$: $d(x,a)>D^{3s}$. Clearly then for all $a\in A$: $d(x,y)\leq D^6\cdot d(x,a)$.

$\boxed{\text{case 2}}$ there is a $b\in A$ such that $d(x,b)<D^3$.
Then we have:

$$(\star)\quad \forall n\in\mathbb{N}\,\exists(s,z)\in\{0,1\}\times A\,[(s=0\wedge d(x,z)<D^{-3n+3})\vee(s=1\wedge\forall a\in A\,[d(x,a)>D^{-3n}]$$

By $\mathbf{AC}_{01}$ there is a function h from $\mathbb{N}$ to $\{0,1\}\times A$ realizing $(\star)$. Define a Cauchy-sequence in (A,d) by putting $y_0=b$ and for $n\in\mathbb{N}$:

$$y_{n+1} \underset{D}{=} \begin{cases} z & \text{if } h(n+1)=(0,z) \\ y_n & \text{else} \end{cases}$$

Then for $y = d\text{-}\lim(y_n)_{n\in\mathbb{N}} \in \overline{(A,d)}$ we have: for all $a \in A$: $d(x,y) \leq D^6 \cdot d(x,a)$.

The above reasoning also simply implies that if (A,d) is located in (X,d) , then $\overline{(A,d)}$ is strongly located in $\overline{(X,d)}$, since $\{D^6 \mid D \in \mathbb{R}_{>1}\} = \mathbb{R}_{>1}$. The implications the other way round follow from our remark above and the fact that (A,d) is dense in $\overline{(A,d)}$ •

EXAMPLE: we give a Brouwerian counterexample to the statement 'if (A,d) is strongly sublocated in (X,d) , then $\overline{(A,d)}$ is strongly traceable in $\overline{(X,d)}$'. Let $A = \{-1\} \cup \{-2^{-n} \mid n \in \mathbb{N} \mid \exists m \in \mathbb{N}\ [m = k_{99}]\}$ and $X = A \cup \{2^{-n} \mid n \in \mathbb{N}\}$. Let $d = d_{\mathbb{R}}$ then (A,d) is strongly sublocated in (X,d) . However $0 \in \overline{X}$, and if we have an α in $\overline{A}$ such that $0 \# \alpha$ implies $\forall \beta \in \overline{A}\ [0 \# \beta]$, then we can decide $\exists m \in \mathbb{N}\ [m = k_{99}]\}$ or $\forall m \in \mathbb{N}\ [m < k_{99}]\}$.

REMARK: notice that the above also is a Brouwerian counterexample to the statement 'if (A,d) is strongly sublocated in (X,d) , then (A,d) is halflocated in (X,d)'. This reveals a disadvantage to the concepts of '(half)located in': these are not topological relations, since it is easy to define a d-equivalent metric d' on (X,d) such that (A,d') is located in (X,d') . However we have theorems 4.5.2 and 4.5.3. A cheap remedy is given in the next definition (following 1.1.2).

DEFINITION: let xdt be a metric space, and let (A,d) be a subspace of (X,d) . Then (A,d) is *topologically (half)located* iff there is a d-equivalent metric d' on (X,d) such that (A,d') is (half)located in (X,d') . (A,d) is *topologically strongly(half)located* iff there is a d-equivalent metric d' on (X,d) such that (A,d') is strongly (half)located in (X,d') . (A,d) is *topologically best approximable* in (X,d) iff there is a d-equivalent metric d' on (X,d) such that (A,d') is best approximable in (X,d') .

3.2.3* the previous example can be sharpened:

EXAMPLE: we give a Brouwerian counterexample to the statement 'if (A,d) is strongly sublocated in (X,d) , then $\overline{(A,d)}$ is traceable in $\overline{(X,d)}$'. Let $A = \{-1\} \cup \{0 \mid \exists n \in \mathbb{N}\ [n = k_{99}]\}$ and $X = A \cup \{2^{-n} \mid n \in \mathbb{N}\}$. Let $d = d_{\mathbb{R}}$, then (A,d) is strongly sublocated in (X,d) . However $0 \in \overline{X}$, and if we can decide: $\exists \alpha \in \overline{A}\ [d(0,\alpha) < 1]$ or $\forall \beta \in \overline{A}\ [0 \# \beta]$, then we can decide $\exists n \in \mathbb{N}\ [n = k_{99}]$ or $\forall n \in \mathbb{N}\ [n < k_{99}]$.

REMARK: by theorem 4.5.2 (ii) there is a d-equivalent metric d' on (X,d) such that (A,d') is strongly halflocated in (X,d').

3.2.4* for complete metric spaces we find however:

LEMMA: let (A,d) be a subspace of a complete metric space (X,d). Then (A,d) is sublocated in (X,d) iff $\overline{(A,d)}$ is strongly sublocated in (X,d).

PROOF: let x be arbitrary in X. We must come up with a y in $\overline{A}$ such that $x\#y$ implies $\exists n\in\mathbb{N}\ \forall a\in\overline{A}\ [d(x,a)>2^{-n}]$. Since (A,d) is sublocated in (X,d) we have:

($\star$) $\forall m\in\mathbb{N}\ \exists(s,a)\in\{0,1\}\times A\ [(s=0\wedge d(x,a)<2^{-m})\vee$
$(s=1\wedge\exists n\in\mathbb{N}\forall a\in A\,[d(x,a)>2^{-n}])]$

By $\mathbf{AC}_{01}$ there is a function h from $\mathbb{N}$ to $\{0,1\}\times A$ realizing ($\star$). Let h_0 and h_1 be functions from $\mathbb{N}$ to $\{0,1\}$ and A respectively such that for all $m\in\mathbb{N}$: $h(m)=(h_0(m),h_1(m))$. We define a Cauchy-sequence $(a_m)_{m\in\mathbb{N}}$ in (A,d) as follows. Put $a_0=h_1(0)$ and for $m\in\mathbb{N}$:

$$a_{m+1}=\begin{cases} h_1(m+1) & \text{if } h_0(m+1)=0\\ a_m & \text{else}\end{cases}$$

Put $y=d\text{-}\lim(a_m)_{m\in\mathbb{N}}\in\overline{A}$. Clearly $x\#y$ implies $\exists n\in\mathbb{N}\ \forall a\in\overline{A}\ [d(x,a)>2^{-n}]$. The implication the other way round follows from remark 3.2.0 and the fact that (A,d) is dense in $\overline{(A,d)}$ •

3.2.5* LEMMA: if (B,d) is (strongly) halflocated in (A,d), and (A,d) is (strongly) halflocated in (X,d), then (B,d) is (strongly) halflocated in (X,d).

PROOF: first let (B,d) be strongly halflocated in (A,d),and (A,d) strongly halflocated in (X,d), with parameters $D,E\in\mathbb{R}_{>1}$ respectively. Let x be in X, determine y in A, z in B such that $\forall a\in A\ [d(x,y)\leq E\cdot d(x,a)]$ and $\forall b\in B\ [d(y,z)\leq D\cdot d(y,b)]$. Now let b be in B. Then $d(x,y)\leq E\cdot d(x,b)$ and $d(y,z)\leq D\cdot d(y,b)]$, therefore $d(x,z)\leq E\cdot d(x,b)+D\cdot d(y,b)$. On the other hand, $d(y,b)\leq d(x,b)+d(x,y)\leq(E+1)\cdot d(x,b)$. So $d(x,z)\leq(E+1)(D+1)\cdot d(x,b)$.

Now suppose we know only that (B,d) is halflocated in (A,d), and (A,d) is halflocated in (X,d). Then by lemma 3.2.2 $\overline{(B,d)}$ is strongly halflocated in $\overline{(A,d)}$ and $\overline{(A,d)}$ is strongly halflocated in $\overline{(X,d)}$. So by the above $\overline{(B,d)}$ is strongly halflocated in $\overline{(X,d)}$, so by lemma 3.2.2 (B,d) is halflocated in (X,d) •

EXAMPLE: we give a Brouwerian counterexample to the statement: 'if (B,d) is strongly located in (A,d), where (A,d) is strongly located in (X,d), then (B,d) is located in (X,d)'. Let $B=\{0\}\cup\{3\,|\,\exists n\in\mathbb{N}\,[n=k_{99}]\}$, $A=B\cup\{1\}$ and $X=A\cup\{2\}$. Put $d(0,2)=2$ and for $i<j\leq 3$, $i+j\neq 2$: $d(i,j)=1$.

REMARK: notice that $(B,d),(A,d)$ and (X,d) are all complete metric spaces. It is not difficult to lead the above dubious statement to a contradiction by using **CP**. The example reveals another disadvantage to the concept of 'located in'. For even were we to topologize the concept by defining: '(A,d) is topologically located in (X,d) iff there is a d-equivalent metric d' on X such that (A,d') is located in (X,d')', then still we would be in the dark as to the transitivity of 'topologically located in'. Notice that, in accordance with the lemma, (B,d) is strongly halflocated in (X,d), so the example is a Brouwerian counterexample to the statement 'if (B,d) is strongly halflocated in (X,d), then (B,d) is located in (X,d)', which in the same way leads to a contradiction using **CP**.

3.2.6* in chapter one, lemma 1.3.4 we showed that if $(B,\mathcal{T}_B)$ is strongly sublocated in $(A,\mathcal{T}_A)$ and $(A,\mathcal{T}_A)$ is strongly sublocated in $(X,\mathcal{T})$, then $(B,\mathcal{T}_B)$ is strongly sublocated in $(X,\mathcal{T})$. So 'strongly sublocated in' and 'strongly traceable in' behave transitively. We mention once more example 1.3.4 (with a little more precision):

EXAMPLE: we give a Brouwerian counterexample to the statement 'if (B,d) is sublocated in (A,d), where (A,d) is sublocated in (X,d), then (B,d) is traceable in (X,d)'. Let $E=\overline{\{2^{-n}\,|\,n\in\mathbb{N}\}}$ (w.r.t. $d_{\mathbb{R}}$). Put $B=\{1\}\cup\{e\in E\,|\,\exists n\in\mathbb{N}\,[n=k_{99}]\}$, put $A=B\cup\{3^{-n}\,|\,n\in\mathbb{N}\}$ and $X=A\cup\{0\}$, and let $d=d_{\mathbb{R}}$.

REMARK: once again the dubious statement is contradictory in the presence of **CP**.

3.2.7 PROPOSITION: if (A,d) is strongly traceable in a spreadlike (X,d), then (A,d) is spreadlike. If moreover (X,d) is compact, then (A,d) is compact.

PROOF: without loss of generality X is a spread, σ say. We have:

$(\star)$ $\quad \forall\alpha\in\sigma\;\exists\beta\in\sigma\;[\beta\in A\wedge(\alpha\#\beta\;\rightarrow\;\forall\delta\in A\,[\alpha\#\delta]\,]$

By $\mathbf{AC}_{11}$ there is a spread-function γ from σ to σ realizing $(\star)$. Clearly for all α in σ: $\gamma(\alpha)\in A$, and for all β in A: $\gamma(\beta)\equiv\beta$. Now define d_γ on σ by putting: $d_\gamma(\alpha,\beta)=d(\gamma(\alpha),\gamma(\beta))$, for α, β in σ. Then (σ,d_γ) coincides with (A,d), the homeomorphism given by γ.

Now suppose that (X,d) is compact, then without loss of generality σ is a fan. By the fan theorem **FT**, for every $n\in\mathbb{N}$ the subset $T_n=\{\gamma_{[n]}(\alpha)\,|\,\alpha\in\sigma\}$ of $\overline{\overline{\sigma}}(n)$ is finite, and $\{\overline{b}(n)\,|\,b\in T_{n+1}\}$ equals T_n. Because of this we can define a fan τ as follows:

$\tau(a)=0$ iff $a\in T_{lg(a)}$, for $a\in\mathbb{N}$.

Let α be in τ. Suppose $\alpha\#\gamma(\alpha)$. Then, since γ is a spread-function, there is an $n\in\mathbb{N}$ such that for all β in $\sigma:\overline{\alpha}(n)=\overline{\beta}(n)$ implies $\beta\#\gamma(\beta)$ implies $\forall\delta\in A\ [\beta\#\delta]$. Contradiction, for clearly for all $n\in\mathbb{N}$ there is a $\beta\in\sigma\cap\overline{\alpha}(n)\cap A$. Therefore $\alpha\equiv\gamma(\alpha)$, and so α is in A. On the other hand, if α is in A, then $\alpha\equiv\gamma(\alpha)\in\tau$. So (A,d) coincides with (τ,d) •

3.2.8 PROPOSITION: let (A,d) be traceable in a compact metric space (X,d). Then (A,d) is located in (X,d).

PROOF: without loss of generality X is a fan, say τ. Let $n\in\mathbb{N}$, then we have:

$$(\star)\qquad \forall\alpha\in\tau\ \exists s\in\{0,1\}\ [\,(s=0\wedge\exists\beta\in A\,[d(\alpha,\beta)<2^{-n})]\vee(s=1\wedge\forall\beta\in A\,[\alpha\#\beta])\,]$$

Then by the fan theorem **FT** there is an $m\in\mathbb{N}$ such that: m_1 is a function from $\overline{\overline{\tau}}(m_0)$ to $\{0,1\}$ such that:

$$(\star\star)\qquad \forall\alpha\in\tau\ [\,m_1(\overline{\alpha}(m_0))\text{ realizes }(\star)\text{ for }\alpha\,]$$

So for all $n\in\mathbb{N}$ there is an $m\in\mathbb{N}$ realizing $(\star\star)$ for n. Then by $\mathbf{AC}_{00}$ there is a function h from $\mathbb{N}$ to $\mathbb{N}$ such that for all $n\in\mathbb{N}$ $h(n)$ realizes $(\star\star)$ for n. Notice that for all $n\in\mathbb{N}$: $\exists a\in\overline{\overline{\tau}}((h(n))_0)\ [(h(n))_1(a)=0]$, since A is inhabited. For $n\in\mathbb{N}$ put $\tau_n=\{\alpha\in\tau\cap a\,|\,a\in\overline{\overline{\tau}}((h(n))_0)\wedge(h(n))_1(a)=0\}$. Then for all $n\in\mathbb{N}$ τ_n is a subfan of τ and $A\subseteq\tau_n$. Let α be in τ.

$\boxed{\text{claim}}$ $\inf(\{d(\alpha,\beta)\,|\,\beta\in A\}=d_{\mathbb{R}}\text{-lim}(\inf(\{d(\alpha,\gamma)\,|\,\gamma\in\tau_n\})_{n\in\mathbb{N}}$.

$\boxed{\text{proof}}$ by the fan theorem **FT** we can compute $\rho_n = \inf(\{d(\alpha,\gamma) \mid \gamma \in \tau_n\})$ for each $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $a \in \overline{\overline{\tau}}((h(n))_0)$. Then there is a $\beta \in A$ such that $d(\alpha_a, \beta) < 2^{-n}$. So there is a $\beta \in A$ such that $d(\alpha,\beta) < \rho_n + 2^{-n}$. But of course $\overline{\beta}((h_{n+1})_0)) \in \overline{\overline{\tau}}_{n+1}$ so $\rho_{n+1} < \rho_n + 2^{-n}$. Also, for all $\beta \in A$: $d(\alpha_a, \beta) \geq \rho_n$, since $A \subseteq \tau_n$. Therefore $\rho_{n+1} > \rho_n - 2^{-n}$. (For suppose $\rho_{n+1} < \rho_n - 2^{-n-1}$. Then there is a δ in A such that $d(\alpha,\delta) < \rho_{n+1} + 2^{-n-1} < \rho_n$. Contradiction.). So $(\rho_n)_{n\in\mathbb{N}}$ is $d_{\mathbb{R}}$-Cauchy, and the rest of the claim follows trivially ∘ •

The following examples however show that for definition 3.2.1 the situation is more complicated.

EXAMPLE: we give a Brouwerian counterexample to the statement 'if (A,d) is strongly sublocated in a compact (X,d), then (A,d) is strongly halflocated in (X,d)'. Let $A = \{0,2\}$, let $X = \{0,1,2\}$ and define $\rho_{0,1}, \rho_{1,2} \in \mathbb{Q}^{\mathbb{N}}_{\geq 0}$ by putting, for $n \in \mathbb{N}$:

$\rho_{0,1}(n) = 0$	$\rho_{1,2}(n) = 0$	if $n < k_{99}$
$\rho_{0,1}(n) = \frac{1}{k_{99}}$	$\rho_{1,2}(n) = \frac{1}{k_{99}^2}$	if $k_{99} \leq n$ and k_{99} is even
$\rho_{0,1}(n) = \frac{1}{k_{99}^2}$	$\rho_{1,2}(n) = \frac{1}{k_{99}}$	if $k_{99} \leq n$ and k_{99} is odd

Define: $d(0,1) = d_{\mathbb{R}}\text{-}\lim(\rho_{0,1}(n))_{n\in\mathbb{N}}$, $d(1,2) = d_{\mathbb{R}}\text{-}\lim(\rho_{1,2}(n))_{n\in\mathbb{N}}$ and $d(0,2) = d(0,1) + d(1,2)$. Replace k_{99}^2 by $2k_{99}$ to get a Brouwerian counterexample to the statement: 'if (A,d) is strongly halflocated in a compact (X,d), then (A,d) is strongly located in (X,d)'.

EXAMPLE: we give a Brouwerian counterexample to the statement 'if (A,d) is strongly sublocated in a 1-locally strongly compact space (X,d), then (A,d) is strongly halflocated in (X,d)'. Let $A = \{1\} \cup \{2^{-n} \mid n = k_{99}\}$, let $X = (0,1]$ and let $d = d_{\mathbb{R}}$. Replace $X = (0,1]$ by $X = [\frac{1}{3}, 1] \cup \{2^{-n} \mid n = k_{99}\}$ to get a Brouwerian counterexample to the statement 'if (A,d) is strongly halflocated in a 1-locally strongly compact space (X,d), then (A,d) is strongly located in (X,d)'.

3.2.9 but for a boundedly strongly compact space (definition 2.2.7), such as $(\mathbb{R}, d_{\mathbb{R}})$, we have:

THEOREM: let (A,d) be (strongly) traceable in a boundedly strongly compact space (X,d). Then (A,d) is (strongly) located in (X,d).

PROOF: since (X,d) is complete, by theorem 3.0.2 we can let i be an isometry from (X,d) to (σ,d), a metric spread. First suppose (A,d) is strongly traceable in (X,d). By lemma

3.3.13 (A,d) is strongly sublocated in (X,d), and therefore complete by remark 3.2.1. Now let α be in σ, determine β in $i(A)$ such that $\alpha\#\beta$ implies $\forall\delta\in i(A)\,[\alpha\#\delta]$. Put $\epsilon=d(\alpha,\beta)$, then we have that $\overline{(\{\alpha\}\cup B(\alpha,\epsilon),d)}$ is a compact subspace of (X,d), which therefore coincides with a subfan (ρ,d) of (σ,d). Let γ be a spread-function from σ to σ constructed as in the proof of proposition 3.2.7. By the same argument as in that proof, for every $n\in\mathbb{N}$ the set $T_n=\{\gamma_{[n]}(\delta)\,|\,\delta\in\rho\}$ is finite. We can therefore define a fan τ as follows:

$\tau(a)=0$ iff a is in $T_{lg(a)}$, for a in $\mathbb{N}$.

By lemma 0.4.3, $\overline{(\tau,d)}$ is located in $\overline{(\tau,d)}\cup(\rho,d)$, and by lemma 3.2.2 $\overline{(\tau,d)}$ is strongly located in $\overline{(\tau,d)}\cup(\rho,d)$. But $\overline{(\tau,d)}\subseteq i(A)$ since $(i(A),d)$ is complete. Now let D be in $\mathbb{R}_{>1}$. Determine y in $\overline{(\tau,d)}\cup(\rho,d)\subseteq i(A)$ such that for all δ in $\overline{(\tau,d)}\cup(\rho,d)$: $d(\alpha,y)\leq D\cdot d(\alpha,\delta)$. But then $d(\alpha,y)\leq D\cdot d(\alpha,\delta)$ for all δ in σ. Since D is arbitrary, $(i(A),d)$ is strongly located in (σ,d).

Now suppose we know only that (A,d) is traceable in (X,d). Then by remark 3.2.1 we have that $\overline{(A,d)}$ is strongly traceable in (X,d). By the above reasoning $\overline{(A,d)}$ is strongly located in (X,d). Therefore by lemma 3.2.2 (A,d) is located in (X,d) •

COROLLARY: let (A,d) be a (strongly) traceable in a 1-locally strongly compact (X,d). Then (A,d) is (strongly) topologically located in (X,d).

PROOF: by the second corollary in 2.2.7 there is a d-equivalent metric d' on (X,d) such that (X,d') is boundedly strongly compact. By the theorem (A,d') is (strongly) located in (X,d') •

REMARK: the theorem contradicts [Troelstra&vanDalen88, p.360, l.34] which promises 'a counterexample in $\mathbb{R}^2$' to the statement: 'if (A,d) is sublocated in (X,d), then (A,d) is located in (X,d)'. The connected exercise 7.3.2 mentions only $\mathbb{N}$, and is as follows. For β in σ_2 an injection ϕ from $(\mathbb{N},d_{\mathbb{R}})$ to $(\mathbb{N},d_{\mathbb{R}})$ is given thus: $\phi(n)=1$ if $n=\mu t\in\mathbb{N}\,[\beta(n)=0]$ and $\phi(n)=n+2$ else. Now if $A=\{n\in\mathbb{N}|n\geq 3\}$, then $(A,d_{\mathbb{R}})$ is located in $(\mathbb{N},d_{\mathbb{R}})$, but $(\phi(A),d_{\mathbb{R}})$ is located in $(\phi(\mathbb{N}),d_{\mathbb{R}})$ iff $\beta\#_\omega\underline{1}\vee\beta=\underline{1}$. Observe that this is *not* a counterexample in $(\mathbb{N},d_{\mathbb{R}})$, but a (Brouwerian) counterexample in $(\phi(\mathbb{N}),d_{\mathbb{R}})$. Of course we can also define a metric d_ϕ on $\mathbb{N}$ such that (A,d_ϕ) is located in $(\mathbb{N},d_\phi)$ iff $\beta\#_\omega\underline{1}\vee\beta=\underline{1}$. But then we have a counterexample in $(\mathbb{N},d_\phi)$, *not* in $(\mathbb{N},d_{\mathbb{R}})$. So the confusion in [Troelstra&vanDalen88] is probably due to the relevant metrics' not being mentioned explicitly.

3.3 WEAK STABILITY

3.3.0* DEFINITION: let (A,d) be a subspace of a metric space (X,d).

(i) (A,d) is *stable* in (X,d) iff for all x in X: $\neg\neg\exists a\in A\ [x\equiv a]$ implies $\exists a\in A\ [x\equiv a]$, that is $x\in A$.

(ii) (A,d) is *weakly stable* in (X,d) iff for all x in X: $\exists a\in A\ [x\# a \to x\in A]$ implies $x\in A$.

REMARK: if (A,d) is (weakly) stable in (X,d), and h is a homeomorphism from (X,d) to (Y,d_Y), then $(h(A),d_Y)$ is (weakly) stable in (Y,d_Y). Also, if (A,d) is (weakly) stable in (B,d) and (B,d) is (weakly) stable in (X,d), then (A,d) is (weakly) stable in (X,d).

For instance $(\mathbb{R}_{\geq 0},d_{\mathbb{R}})$ is stable in $(\mathbb{R},d_{\mathbb{R}})$, and $(\neg\neg\mathbb{Q},d_{\mathbb{R}})$ is stable in $(\mathbb{R},d_{\mathbb{R}})$, with $\neg\neg\mathbb{Q}=\{\alpha\in\mathbb{R}\mid\neg\neg\exists q\in\mathbb{Q}\ [\alpha\equiv q]\}$. There are two main problems with the notion of stability. Firstly, for any (A,d) we can define the stable closure of (A,d) in (X,d) as $(\{x\in X\mid\neg\neg\exists a\in A\ [x\equiv a]\},d)$. But there is hardly an effective notion of universal stability for a given metric space (A,d), in the sense that such an (A,d) would be stable in any (X,d) in which it is contained as a subspace (except if (A,d) is strongly compact, then (A,d) is stable in any (Y,d_Y) in which it is homeomorphically contained). To ask for a topological notion of universal stability is more difficult still. As an example consider $(\mathbb{R},d_{\mathbb{R}})$, which is almost as nice a space as one could wish for. However $((0,1),d_{\mathbb{R}})$ is homeomorphic to $(\mathbb{R},d_{\mathbb{R}})$, yet saying that $((0,1),d_{\mathbb{R}})$ is stable in $([0,1],d_{\mathbb{R}})$ is equivalent to Markov's Principle. So we cannot even prove that 'stable in its own completion' is a topological property (except if (X,d) is strongly compact).

The second problem is that given a metric spread (ρ,d) which is a subspace of a metric spread (σ,d), we do not see a way to construct the stable closure of (ρ,d) in (σ,d) as a metric spread, which would seem desirable.

In the course of our investigations in chapter four we ran into this question of stability. Given the difficulties described, we sought to weaken the notion of stability, rather than limit our theorems to complete metric spaces. We came to weak stability, which we now believe to be a fruitful concept. Firstly the above mentioned difficulties can all be solved, secondly weak stability suffices for what we do in chapter four.

Thirdly, for example, [Veldman&Waaldijk96, thm. 3.3.4] states that a stable dense subset A of $(\mathbb{R}, d_{\mathbb{R}})$ gives rise to an elementary substructure $\langle A, \#_{\mathbb{R}}\rangle$ of $\langle \mathbb{R}, \#_{\mathbb{R}}\rangle$. The proof given in fact requires only that A be dense in $(\mathbb{R}, d_{\mathbb{R}})$ and weakly stable.

Finally, weak stability led us to the Complete Metric Continuity Principle ($\mathbf{CP}_{\mathrm{cm}}$), which is derivable from $\mathbf{CP}$, and vice versa, without using $\mathbf{AC}_{10}$. $\mathbf{CP}_{\mathrm{cm}}$ by itself extends Brouwer's theorem on the continuity of everywhere defined real functions. Using $\mathbf{AC}_{10}$ we generalize $\mathbf{CP}_{\mathrm{cm}}$ to the Weakly Stable Continuity Principle ($\mathbf{CP}_{\mathrm{ws}}$), extending Brouwer's theorem even further.

3.3.1* DEFINITION: let (X, d) be a metric space. Then (X, d) is *weakly stable* iff (X, d) is weakly stable in $\overline{(X, d)}$, the completion of (X, d).

In the rest of this chapter we investigate and prove some fundamental properties of weak stability.

3.3.2* DEFINITION: for each metric space (X, d) we define the *weakly stable closure* of (X, d), notation $\overset{\cdots}{(X, d)}$, as a subspace of $\overline{(X, d)}$. Let $W_0(X, d) \underset{D}{=} (X, d)$ and for $n \in \mathbb{N}$ put $W_{n+1}(X, d) \underset{D}{=} \{\alpha \in \overline{X} \mid \exists \beta \in W_n(X, d)\ [\alpha \# \beta \rightarrow \alpha \in W_n(X, d)]\}$. Now we define: $\overset{\cdots}{(X, d)} \underset{D}{=} (\bigcup_{n \in \mathbb{N}} W_n(X, d), d)$. We will frequently write $\overset{\cdots}{X}$ for $\bigcup_{n \in \mathbb{N}} W_n(X, d)$ in situations where it is clear to which metric we're referring.

REMARK: trivially $\overset{\cdots}{(X, d)}$ is weakly stable, and if (X, d) is weakly stable then $\overset{\cdots}{(X, d)}$ coincides isometrically with (X, d).

EXAMPLE: let $X = \{\alpha \in \sigma_{2\mathrm{mon}} \mid \alpha \equiv \underline{0} \vee \alpha \# \underline{0}\}$, and $d = d_\omega$. Notice that $\overline{(X, d)}$ is isometrically homeomorphic to $(\sigma_{2\mathrm{mon}}, d_\omega)$. Now $\alpha_{k_{99}} \in \sigma_{2\mathrm{mon}}$ and $\alpha_{k_{99}} \# \underline{0}$ implies $\alpha_{k_{99}} \in X$. But to say that $\alpha_{k_{99}} \in X$ is daring. Using $\mathbf{CP}$ we can easily prove: (X, d) is NOT weakly stable. Notice that $W_1(X, d)$ coincides with $(\sigma_{2\mathrm{mon}}, d_\omega)$, so $\overset{\cdots}{(X, d)}$ coincides with $\overline{(X, d)}$, which is a rare situation indeed for a non-complete space. Similarly $(\mathbb{Q}, d_{\mathbb{R}})$ is NOT weakly stable, but for $n \in \mathbb{N}$ neither is $(W_n(\mathbb{Q}, d_{\mathbb{R}}), d_{\mathbb{R}})$. We do have that $\overset{\cdots}{(\mathbb{Q}, d_{\mathbb{R}})}$ coincides with a metric spread (see theorem 3.3.9 and remark 3.0.1).

3.3.3* EXAMPLE: consider $[0, 1]_3$, the ternary real numbers in $[0, 1]$. Let $(a_n)_{n \in \mathbb{N}}$ be a Cauchy-sequence in $([0, 1]_3, d_{\mathbb{R}})$ given by:

$$a_n \underset{D}{=} \begin{cases} 3^{-1} & \text{if } n < k_{99} \\ 3^{-1} + (-3)^{-k_{99}} & \text{else} \end{cases}$$

Let $\alpha = d_{\mathbb{R}}\text{-}\lim(a_n)_{n\in\mathbb{N}}$, then $\alpha \# \frac{1}{3}$ implies $\alpha \in [0,1]_3$. But it is daring to say that $\alpha \in [0,1]_3$, since then we can decide: k_{99} is even or k_{99} is odd, by just looking at the first digit of β in $[0,1]_3$ such that $\beta \equiv \alpha$. This shows that it is daring to say that $([0,1]_3, d_{\mathbb{R}})$ is weakly stable. Using **CP** we can prove: $([0,1]_3, d_{\mathbb{R}})$ is NOT weakly stable (see 3.3.14).

The question now arises whether $\overset{\cdots}{[0,1]_3}$ in fact equals $[0,1]$. Using $\mathbf{AC}_{11}$ and **FT** we can prove: $\neg \forall \alpha \in [0,1] \ \exists \beta \in \overset{\cdots}{[0,1]_3} \ [\alpha \equiv \beta]$ (see 3.3.14). Then: $[0,1]_3 \subsetneq \overset{\cdots}{[0,1]_3} \subsetneq [0,1]$, so we have found an interesting space in between $([0,1]_3, d_{\mathbb{R}})$ and $([0,1], d_{\mathbb{R}})$. We will show in 3.3.14 that $\overset{\cdots}{([0,1]_3, d_{\mathbb{R}})}$ is a sigma-compact space which is NOT locally compact.

3.3.4* LEMMA: let (X,d) be a metric space, and let $n \in \mathbb{N}$. Suppose $y_1, y_2, \ldots, y_n$ are in X , and x is in $\overline{(X,d)}$ such that: $\forall i \in \{1,\ldots,n\} \ [x \# y_i]$ implies $x \in X$. Then x is in $W_n(X,d)$.

PROOF: we prove the lemma for all $n \in \mathbb{N}$ by induction.
Basis: $n=0$. Then the lemma is trivially true.
Induction: let $n \in \mathbb{N}$ such that that the lemma holds true for n . We show that the lemma holds for $n+1$. Let $y_1, y_2, \ldots, y_{n+1}$ be in X , and x in $\overline{(X,d)}$ such that: $\forall i \in \{1,\ldots,n+1\} \ [x \# y_i]$ implies $x \in X$. Clearly then $x \# y_{n+1}$ implies that: $\forall i \in \{1,\ldots,n\} \ [x \# y_i]$ implies $x \in X$. By induction this gives: $x \# y_{n+1}$ implies $x \in W_n(X,d)$. Since y_{n+1} is in $W_n(X,d)$ this gives that $x \in W_{n+1}(X,d)$ •

3.3.5* LEMMA:

(i) let f be a continuous function from (X,d) to a metric space (Y, d_Y) . Then there is a continuous function $\tilde{f}$ from $W_1(X,d)$ to $W_1(Y,d_Y)$ such that the restriction of $\tilde{f}$ to (X,d) coincides with f .

(ii) if i is an injection (homeomorphism) of (X,d) into (Y,d_Y) , then we can extend i to an injection (homeomorphism) of $W_1(X,d)$ into $W_1(Y,d_Y)$.

PROOF: For (i) let $f : (X,d) \longmapsto (Y,d_Y)$ be continuous. Let $\beta \in W_1(X,d)$ and determine $\alpha \in X$ such that $\beta \# \alpha$ implies $\beta \in X$. Determine a sequence $(\delta_m)_{m\in\mathbb{N}}$ in $\mathbb{R}^+$ such that for all $m \in \mathbb{N}$: $\delta_{m+1} < \frac{1}{2}\delta_m$ and for all γ in X: $d(\gamma,\alpha) < \delta_m$ implies $d_Y(f(\gamma), f(\alpha)) < 2^{-m}$. We find:

$$(\star) \qquad \forall m \in \mathbb{N} \ \exists s \in \{0,1\} \ [\, (s=0 \wedge d(\beta,\alpha) < \delta_m) \ \vee \ (s=1 \wedge d(\beta,\alpha) > \tfrac{1}{2}\delta_m) \,]$$

By $\mathbf{AC}_{00}$ there is a function h from $\mathbb{N}$ to $\{0,1\}$ realizing $(\star)$. Define a Cauchy-sequence in (Y,d_Y) as follows:

$$z_s \underset{D}{=} \begin{cases} f(\alpha) & \text{if } h(s)=0 \\ f(\beta) & \text{if } h(s)=1 \end{cases}$$

Put $\tilde{f}(\beta) \underset{D}{=} d\text{-}\lim(z_s)_{s\in\mathbb{N}}$. Then clearly $\tilde{f}(\beta)\#f(\alpha)$ implies: $\tilde{f}(\beta)\in(Y,d_Y)$. Therefore $\tilde{f}(\beta)$ is in $W_1(Y,d_Y)$. To prove that $\tilde{f}$ is continuous, let $m\in\mathbb{N}$.

$\boxed{\text{case 1}}$ $d(\beta,\alpha)<\delta_{m+1}$

Then let $\gamma\in W_1(X,d)$ such that $d(\beta,\gamma)<\delta_{m+1}$. Then $d(\gamma,\alpha)<\delta_m$. Observe that $\neg\neg(\gamma\in X)$ so $\neg\neg(d_Y(\tilde{f}(\gamma),f(\alpha))<2^{-m})$ from which we obtain that $d_Y(\tilde{f}(\gamma),\tilde{f}(\beta))\leq 2^{-m+1}$

$\boxed{\text{case 2}}$ $d(\beta,\alpha)>\frac{1}{2}\delta_{m+1}$

Then $\beta\in X$ so we can find $\delta\in\mathbb{R}^+$ such that for all $y\in X$: $d(y,\beta)<\delta$ implies $d_Y(f(y),\tilde{f}(\beta)<2^{-m+1}$. By the same double negation reasoning as in case 1 we find that if $\gamma\in W_1(X,d)$ such that $d(\beta,\gamma)<\delta$, then $d_Y(\tilde{f}(\gamma),\tilde{f}(\beta))\leq 2^{-m+1}$.

m being arbitrary, $\tilde{f}$ is continuous.

Finally for (ii) we use (i) to construct an extension $\tilde{\imath}$ of i, from $W_1(X,d)$ to $W_1(Y,d_Y)$. Clearly $\tilde{\imath}$ is injective. Now suppose i is a homeomorphism. To show that $\tilde{\imath}$ is surjective, let z be in $W_1(Y,d_Y)$ and determine w in Y such that $z\#w$ implies $z\in Y$. Then we have:

$$(\star) \quad \forall n\in\mathbb{N}\ \exists s\in\{0,1\}\ [\,(s=0 \wedge d(z,w)<2^{-n}) \ \vee\ (s=1 \wedge d(y,w)>2^{-n-1})\,]$$

By $\mathbf{AC}_{00}$ there is a function h from $\mathbb{N}$ to $\{0,1\}$ realizing $(\star)$. Define a Cauchy-sequence $(x_n)_{n\in\mathbb{N}}$ in (X,d) as follows:

$$x_n \underset{D}{=} \begin{cases} i^{-1}(w) & \text{if } h(n)=0 \\ i^{-1}(z) & \text{if } h(n)=1 \end{cases}$$

Notice that $\forall n\in\mathbb{N}\,[\,x_n\in X\,]$. Since i^{-1} is continuous in w, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in (X,d), say with d-limit $\alpha\in\overline{(X,d)}$. We have that $\alpha\#i^{-1}(w)$ implies $\alpha\in X$, so α is in $W_1(X,d)$. But of course $\tilde{\imath}(\alpha)\equiv z$ •

THEOREM:

(i) let f be a continuous function from (X,d) to a metric space (Y,d_Y). Then there is a continuous function $\tilde{f}$ from $\overline{(X,d)}$ to $\overline{(Y,d_Y)}$ such that the restriction of $\tilde{f}$ to (X,d) coincides with f.

(ii) if i is an injection (homeomorphism) of (X,d) into (Y,d_Y), then we can extend i to an injection (homeomorphism) of $\overline{(X,d)}$ into $\overline{(Y,d_Y)}$.

(iii) 'weakly stable' is a topological property.

(iv) let i be an injection of a weakly stable (X,d) into (Y,d_Y). Then $(i(X),d_Y)$ is weakly stable in (Y,d_Y).

PROOF: for (i) and (ii) use the previous lemma (i) and (ii) inductively to define the desired extension on $W_n(X,d)$ for each $n\in\mathbb{N}$. Next, (iii) follows from (ii): let (X,d) be a weakly stable space, and let i be a homeomorphism of (X,d) to (Y,d_Y), with inverse j. Then by (ii) we can extend j to a homeomorphism $\tilde{\jmath}$ of $\overline{(Y,d_Y)}$ to (X,d). Then $i\circ\tilde{\jmath}$ is a homeomorphism from $\overline{(Y,d_Y)}$ to (Y,d_Y), which restricts to the identity on (Y,d_Y). So $\overline{(Y,d_Y)}$ coincides identically with (Y,d_Y), meaning (Y,d_Y) is weakly stable. Finally (iv): let y be in Y, and z in $i(X)$ such that $y\#z$ implies $y\in i(X)$. We have:

$$(\star)\qquad \forall n\in\mathbb{N}\ \exists s\in\{0,1\}\ [\,(s=0\wedge d_Y(y,z)<2^{-n})\ \vee\ (s=1\wedge d_Y(y,z)>2_{-n-1}\,]$$

By $\mathbf{AC}_{00}$ there is a function h from $\mathbb{N}$ to $\mathbb{N}$ realizing ($\star$). Define a Cauchy-sequence in $(i(X),d_Y)$ as follows:

$$w_n \underset{D}{=} \begin{cases} z & \text{if } h(n)=0 \\ y & \text{if } h(n)=1 \end{cases}$$

Put $w=d_Y\text{-}\lim(w_n)_{n\in\mathbb{N}}\in\overline{(i(X),d_Y)}$. Then $w\#z$ implies $w\in i(X)$, and by (iii) $(i(X),d_Y)$ is weakly stable, so $w\in i(X)$. But $y\equiv w$, so $y\in i(X)$ •

REMARK: this theorem shows that $\overline{(X,d)}$ deserves the name 'weakly stable closure of (X,d)'. For if i is an injection of (X,d) into a weakly stable space (Y,d_Y), then we can extend i to an injection $\tilde{\imath}$ of $\overline{(X,d)}$ into (Y,d_Y). Also (iv) indeed shows a weakly stable space to be weakly stable in any space in which it is homeomorphically contained, which along with (iii) was promised in 3.3.0.

3.3.6* we wish to prove that the weakly stable closure of a metric spread (σ,d) coincides with a metric spread. One might be tempted to construct $W_1(\sigma,d)$ as a spread derived from $\sigma^{\mathbb{N}}$, somewhat in the following fashion: an α in $\sigma^{\mathbb{N}}$ is in $W_1(\sigma,d)$ if it starts out as a constant sequence β, β, β, ... in which at most one (slight) deviation to another (close to β) constant sequence can occur. This is feasible if we know that d is given by a spread-function. So in general we would have to use $\mathbf{AC}_{10}$ (or change our definition of a metric spread). Since we wish to show that $\mathbf{CP}_{\mathrm{cm}}$ and $\mathbf{CP}$ are derivable from one another (without using $\mathbf{AC}_{10}$!), we adopt a different approach, which when (σ, d) is steady is also valid in Bishop's school.

3.3.7* DEFINITION: let σ be a spread, and let $n \in \mathbb{N}$. Define a spread $S^n(\sigma)$ as follows. Let a in $\mathbb{N}$, then:

$$S^n(\sigma)(a) \underset{D}{=} \begin{cases} 0 & \text{if } b = \langle\!\langle\rangle\!\rangle \text{ or: } lg(b_0) = n+1 \text{ and } \sigma(b_0 \star \langle\!\langle b_1, \ldots, b_{lg(b)-1} \rangle\!\rangle) = 0 \\ 1 & \text{else} \end{cases}$$

$S^n(\sigma)$ is called the n^{th} *speedup* of σ. We write $S(\sigma)$ for $S^1(\sigma)$.

REMARK: Obviously for any $n \in \mathbb{N}$: $(S^n(\sigma), d_\omega)$ coincides with (σ, d_ω). The definition will only serve a technical purpose: sometimes the n^{th} speedup of a non-steady metric spread is steady.

3.3.8* for the next proposition, recall that (σ, d) is called steady iff for all a in $\overline{\overline{\sigma}}$: $lg(a) > 0$ implies $\forall \alpha, \beta \in \sigma \cap a \, [\, d(\alpha, \beta) < 2^{-lg(a)} \,]$. Also recall that for a in $\overline{\overline{\sigma}}$, $\alpha_a = \alpha_{a,\sigma}$ is a canonical element α in σ such that $\overline{\alpha}(lg(a)) = a$ (definition 0.0.3).

PROPOSITION: let (σ, d) be a steady metric spread. Then $(W_1(\sigma, d), d)$ coincides with a steady metric spread (ρ, d). Moreover, if σ is a fan, then ρ is a fan as well.

PROOF: we have:

$$(\star) \quad \forall a, b \in \overline{\overline{\sigma}} \; \forall n \in \mathbb{N} \; \exists s \in \{0,1\} \, [\, (s=0 \wedge d(\alpha_a, \alpha_b) < 2^{-n+2}) \;\vee\; (s=1 \wedge d(\alpha_a, \alpha_b) > 2^{-n+1}) \,]$$

By $\mathbf{AC}_{00}$ there is a function h from $\overline{\overline{\sigma}} \times \overline{\overline{\sigma}} \times \mathbb{N}$ to $\{0,1\}$ realizing $(\star)$. Define a function k from $\overline{\overline{\sigma}} \times \overline{\overline{\sigma}}$ to $\{0,1\}$ as follows:

$$k(a,b) \underset{D}{=} \begin{cases} 0 & \text{if } a = \langle\!\langle\rangle\!\rangle \\ 0 & \text{if } lg(a) = lg(b) \text{ and } h(a, b, lg(a)) = 0 \\ 1 & \text{else} \end{cases}$$

Now let $a, b \in \overline{\overline{\sigma}}$, $lg(a) = lg(b)$. Suppose $k(a,b) = 0$, then $h(a, b, lg(a)) = 0$ so $d(\alpha_a, \alpha_b) < 2^{-lg(a)+2}$. Since (σ, d) is steady this entails: $\forall \alpha \in \sigma \cap b \, [\, d(\alpha_a, \alpha) < 2^{-lg(a)+3} \,]$. Suppose on the other hand that $k(a,b) = 1$, then $d(\alpha_a, \alpha_b) > 2^{-lg(a)+1}$ and so $\forall \alpha \in \sigma \cap b \, [\, d(\alpha_a, \alpha) > 2^{-lg(a)} \,]$. Define a spread τ as follows: let $c \in \mathbb{N}$, then:

$$\tau(c) \underset{D}{=} \begin{cases} 0 & \text{if } \sigma(\langle\!\langle c_0-1,\ldots,c_{lg(c)-1}-1\rangle\!\rangle)=0 \\ 0 & \text{if } c=a\star\langle\!\langle 0\rangle\!\rangle\star b \text{ for some } a,b\in\mathbb{N} \text{ such that} \\ & \sigma(\langle\!\langle a_0-1,\ldots,a_{lg(a)-1}-1\rangle\!\rangle)=0 \text{ and either } b=\langle\!\langle\rangle\!\rangle \text{ or} \\ & lg(b_0)=lg(a),\ k(\langle\!\langle a_0-1,\ldots,a_{lg(a)-1}-1\rangle\!\rangle,b_0)=0 \text{ and} \\ & \sigma(b_0\star\langle\!\langle b_1,\ldots,b_{lg(b)-1}\rangle\!\rangle)=0 \\ 1 & \text{else} \end{cases}$$

Let α be in τ. Define $\alpha_{\text{left}}\in\sigma$ as follows:

$$\alpha_{\text{left}}(n) \underset{D}{=} \begin{cases} \alpha(n)-1 & \text{if } \forall j\leq n\,[\,\alpha(j)\neq 0\,] \\ t & \text{else, where } t=\mu s\in\mathbb{N}\,[\,\sigma(\overline{\alpha_{\text{left}}}(n-1)\star\langle\!\langle s\rangle\!\rangle)=0\,] \end{cases}$$

Define a Cauchy-sequence $(\beta_n)_{n\in\mathbb{N}}$ in (σ,d) by:

$$\beta_n \underset{D}{=} \begin{cases} \alpha_{\text{left}} & \text{if } \forall j\leq n\,[\,\alpha(j)\neq 0\,] \\ b\star\gamma & \text{if } \alpha=a\star\langle\!\langle 0\rangle\!\rangle\star\langle\!\langle b\rangle\!\rangle\star\gamma \text{ for some } a,b\in\mathbb{N},\ lg(a)\leq n,\ \gamma\in\sigma_\omega \end{cases}$$

Put $i(\alpha)=d\text{-lim}(\beta_n)_{n\in\mathbb{N}}\in\overline{(\sigma,d)}$. Clearly $i(\alpha)\#\alpha_{\text{left}}$ implies $i(\alpha)\in\sigma$, so $i(\alpha)$ is in $W_1(\sigma,d)$. To show that i is surjective, let β be an arbitrary element of $W_1(\sigma,d)$. Determine γ in σ such that $\beta\#\gamma$ implies $\beta\in\sigma$. We then have:

$$(\star\star)\quad \forall n\in\mathbb{N}\ \exists(s,\delta)\in\{0,1\}\times\sigma\ [\,(s=0\wedge d(\beta,\gamma)<2^{-n-1}\wedge\delta=\gamma)\ \vee \\ (s=1\wedge d(\beta,\gamma)>2^{-n-2}\wedge\delta\equiv\beta)\,]$$

By $\mathbf{AC}_{01}$ there is a function h from $\mathbb{N}$ to $\{0,1\}\times\sigma$ realizing $(\star\star)$. Define $j(\beta)$ as follows:

$$j(\beta)(n) \underset{D}{=} \begin{cases} \gamma(n)+1 & \text{if } h(n)=(0,\gamma) \\ 0 & \text{if } n=\mu t\in\mathbb{N}\,[\,h(t)\neq(0,\gamma)\,] \\ \overline{\delta}(n-1) & \text{if } n-1=\mu t\in\mathbb{N}\,[\,h(t)\neq(0,\gamma)\,] \text{ and } h(n-1)=(1,\delta) \\ \delta(n-2) & \text{else, where } s=\mu t\in\mathbb{N}\,[\,h(t)\neq(0,\gamma)\,] \text{ and } h(s)=(1,\delta) \end{cases}$$

Clearly $j(\beta)$ is in τ, and $i\circ j(\beta)\equiv\beta$. We turn to (τ,d):

$\boxed{\text{claim}}$ $\forall e\in\overline{\overline{\tau}}\,[\,lg(e)\geq 3\rightarrow\forall\alpha,\beta\in\tau\cap e\,[\,d(\alpha,\beta)<2^{6-lg(e)}\,]$

$\boxed{\text{proof}}$ let e be in $\overline{\overline{\tau}}$, $lg(e)\geq 3$, and let $\alpha,\beta\in\tau\cap e$. We distinguish:

$\boxed{\text{case 1}}$ $\sigma(\langle\!\langle e_0-1\ldots,e_{lg(e)}-1\rangle\!\rangle)=0$

Then, putting $a=\langle\!\langle e_0-1\ldots,e_{lg(e)}-1\rangle\!\rangle$, we have that $i(\alpha)$ and $i(\beta)$ are in $B(\alpha_a,2^{3-lg(a)}$ so $d(\alpha,\beta)<2^{4-lg(a)}=2^{4-lg(e)}$. For this we only use that $lg(e)\geq 1$.

$\boxed{\text{case 2}}$ $e = a\star\lessdot 0\gtrdot$ for some a in $\overline{\overline{\tau}}$, $lg(a)\geq 2$.
Then by case 1: $d(\alpha,\beta) < 2^{4-lg(a)} = 2^{5-lg(e)}$.

$\boxed{\text{case 3}}$ $e = a\star\lessdot 0\gtrdot\star\lessdot b\gtrdot\star c$ for some $a,b,c\in\mathbb{N}$, $lg(a)=lg(b)$.
Then $lg(b\star c) = lg(e-2)$ and $i(\alpha)$, $i(\beta)$ are in $B(\alpha_{b\star c}, 2^{3-lg(b\star c)})$. Therefore $d(\alpha,\beta) < 2^{4-lg(b\star c)} = 2^{6-lg(e)}$ ∘

Finally we define $\rho \underset{D}{=} S^7(\tau)$, the seventh speedup of τ. By the claim it is clear that (ρ, d) is steady, and of course (ρ, d) coincides isometrically with $(W_1(\sigma,d),d)$. If σ is a fan, then it is easy to see that ρ is a fan as well •

COROLLARY: if (X,d) is compact, then $(W_1(X,d),d)$ is compact.

REMARK: notice that we can now derive from h a canonical function $\tilde{h}$ from $\overline{\overline{\tau}}\times\overline{\overline{\tau}}\times\mathbb{N}$ such that for all (a,b,n) in $\overline{\overline{\tau}}\times\overline{\overline{\tau}}\times\mathbb{N}$: $\tilde{h}(a,b,n)=0$ implies $d(\alpha_{a,\tau},\alpha_{b,\tau}) < 2^{-n+2}$, whereas $\tilde{h}(a,b,n)=1$ implies $d(\alpha_{a,\tau},\alpha_{b,\tau}) > 2^{-n+1}$. This is due to the following:

$\boxed{\text{claim}}$ there is a canonical function g from $\overline{\overline{\tau}}$ to $\overline{\overline{\sigma}}$ such that for all e in $\overline{\overline{\tau}}$: $\alpha_{e,\tau}\equiv\alpha_{g(e),\sigma}$.

$\boxed{\text{proof}}$ as above we distinguish:

$\boxed{\text{case 1}}$ $\sigma(\lessdot e_0 - 1\ldots, e_{lg(e)}-1\gtrdot)=0$.
Then $\alpha_{e,\tau}(lg(e))=0$! Put $a = \lessdot e_0 - 1\ldots, e_{lg(e)}-1\gtrdot$. Since $k(a,a)=0$ we can now determine $b=\mu t\in\overline{\overline{\sigma}}(lg(a))\ [k(a,b)=0]$. Clearly $\alpha_{e,\tau}\equiv\alpha_{b,\sigma}$, so we can put $g(e)=b$.

$\boxed{\text{case 2}}$ $e = a\star\lessdot 0\gtrdot$ for some a in $\overline{\overline{\tau}}$.
Then as in case 1 we determine $b=\mu t\in\overline{\overline{\sigma}}(lg(a))\ [k(a,b)=0]$. Clearly $\alpha_{e,\tau}\equiv\alpha_{b,\sigma}$, so we can put $g(e)=b$.

$\boxed{\text{case 3}}$ $e = a\star\lessdot 0\gtrdot\star\lessdot b\gtrdot\star c$ for some $a,b,c\in\mathbb{N}$.
Clearly $\alpha_{e,\tau}\equiv\alpha_{b,\sigma}$, so we can put $g(e)=b$ ∘

Now we can define, for (a,b,n) in $\overline{\overline{\tau}}\times\overline{\overline{\tau}}\times\mathbb{N}$: $\tilde{h}(a,b,n)=h(g(a),g(b),n)$.

From here it is a triviality to canonically derive from h a function h_1 from $\overline{\overline{\rho}}\times\overline{\overline{\rho}}\times\mathbb{N}$ such that for all (a,b,n) in $\overline{\overline{\rho}}\times\overline{\overline{\rho}}\times\mathbb{N}$: $\tilde{h}(a,b,n)=0$ implies $d(\alpha_{a,\rho},\alpha_{b,\rho}) < 2^{-n+2}$, whereas $\tilde{h}(a,b,n)=1$ implies $d(\alpha_{a,\rho},\alpha_{b,\rho}) > 2^{-n+1}$.

This remark shows that if we choose h, then this proposition can be applied inductively without having to choose an h_1, h_2 ... etc., since these functions can be derived canonically from h.

3.3.9* DEFINITION: let (σ, d) be a steady metric spread. Let h be a function from $\overline{\overline{\sigma}} \times \overline{\overline{\sigma}} \times \mathbb{N}$ realizing $(\star)$ in the proof of proposition 3.3.8. By induction we define, for each $n \in \mathbb{N}$, a spread $\Sigma_n^h(\sigma, d) = \Sigma_n(\sigma, d)$ as follows. Put $\Sigma_0(\sigma, d) \underset{D}{=} \sigma$ and $\Sigma_1(\sigma, d) \underset{D}{=} \rho$, where ρ is as defined in the proof of proposition 3.3.8. More generally, for $n \in \mathbb{N}$: $\Sigma_{n+1}(\sigma, d) \underset{D}{=} \Sigma_1(\Sigma_n(\sigma, d), d)$. Finally put $\Sigma(\sigma, d) \underset{D}{=} S(\dot{\bigcup}_{n \in \mathbb{N}} \Sigma_n(\sigma, d))$, the speedup of $\dot{\bigcup}_{n \in \mathbb{N}} \Sigma_n(\sigma, d)$ and define d on $\Sigma(\sigma, d)$ in the obvious way.

For the correctness of this definition we rely on proposition 3.3.8, and remark 3.3.8.

THEOREM: let (σ, d) be a steady metric spread. Then $\overline{(\sigma, d)}$ coincides isometrically with $(\Sigma(\sigma, d), d)$. Moreover $(\Sigma(\sigma, d), d)$ is steady.

PROOF: by proposition 3.3.8 it is clear that $(\Sigma(\sigma, d), d)$ coincides isometrically with $(\bigcup_{n \in \mathbb{N}} W_n(\sigma, d), d)$ which by definition equals $\overline{(\sigma, d)}$. Since for each $n \in \mathbb{N}$: $(\Sigma_n(\sigma, d), d)$ is steady, $(\Sigma(\sigma, d), d)$ is steady •

3.3.10 REMARK: let (X, d) be a complete metric space. Then (X, d) coincides isometrically with a steady metric spread (σ, d). This is just theorem 3.0.2.

THEOREM: $(\mathbf{CP}_{\rm cm})$ let (X, d) be a complete metric space. Let A be a subset of $X \times \mathbb{N}$ such that:

(i) $\forall x \in X\ \exists n \in \mathbb{N}\ [(x, n) \in A]$

(ii) $\forall x, y \in X\ \forall n \in \mathbb{N}\ [(x \equiv y \wedge (x, n) \in A) \rightarrow (y, n) \in A]$

Then: $\forall x \in X\ \exists n, m \in \mathbb{N}\ \forall y \in X\ [d(x, y) < 2^{-m} \rightarrow (y, n) \in A]$.

PROOF: let (σ, d) be a steady metric spread such that (X, d) coincides isometrically with (σ, d). Clearly (σ, d) is weakly stable. Now follow the proof of theorem 3.3.12 •

COROLLARY: every complete metric space is an apartness space.

COROLLARY: let f be a weak function from a complete metric space (X, d) to another topological space $(Y, \mathcal{T})$. Then f is a function, which in addition is continuous.

REMARK: for a nice proof of the corollaries, if you need one, see 3.3.12. Notice that $\mathbf{CP}_{\text{cm}}$ implies $\mathbf{CP}$, and that in proving $\mathbf{CP}_{\text{cm}}$ we have used only $\mathbf{CP}$. Therefore $\mathbf{CP}_{\text{cm}}$, although seemingly much stronger, is actually equivalent to $\mathbf{CP}$ (whereas we use $\mathbf{AC}_{10}$ to prove theorem 3.3.12 ($\mathbf{CP}_{\text{ws}}$)). Also, this result generalizes a part of Brouwer's famous theorem on the continuity of everywhere defined real functions. Brouwer proved that every function from $([0,1], d_{\mathbb{R}})$ to $(\mathbb{R}, d_{\mathbb{R}})$ is uniformly continuous, using the fan theorem $\mathbf{FT}$, see [Brouwer27]. In [Veldman82] it is proved, using only $\mathbf{CP}$, that every function from $([0,1], d_{\mathbb{R}})$ to $(\mathbb{R}, d_{\mathbb{R}})$ is continuous. The proof relies on the construction of the reals as sequences of strictly shrinking open rational intervals. This technique can be carried out for an arbitrary complete metric space, to also give $\mathbf{CP}_{\text{cm}}$. We find it interesting, however, to give a different and broader approach.

3.3.11 now let's look at a metric spread (σ, d) which is not necessarily steady. We wish to construct the weakly stable closure of (σ, d) as a metric spread. It turns out we can speed up σ in a grand fashion, using $\mathbf{AC}_{10}$:

LEMMA: let (σ, d) be a metric spread, then (σ, d) coincides isometrically with a steady metric spread.

PROOF: let $n \in \mathbb{N}$, then an easy application of $\mathbf{CP}$ gives us:

$$(\star) \quad \forall \alpha \in \sigma\ \exists m \in \mathbb{N}\ \forall \beta \in \sigma\ [\,\overline{\beta}(m) = \overline{\alpha}(m)\ \rightarrow\ d(\alpha, \beta) < 2^{-n}\,]$$

By $\mathbf{AC}_{10}$ there is a spread-function $\gamma \in \sigma_\omega$ realizing $(\star)$. Since n is arbitrary we find:

$$(\star\star) \quad \forall n \in \mathbb{N}\ \exists \gamma \in \sigma_\omega\ [\,\gamma \in \mathit{Fun} \wedge \gamma \text{ realizes } (\star)\,]$$

So by $\mathbf{AC}_{01}$ we obtain a sequence $(\gamma_n)_{n \in \mathbb{N}}$ of spread-functions such that for each $n \in \mathbb{N}$ γ_n realizes $(\star)$. Define a spread τ as follows. Let a be in $\mathbb{N}$, then:

$$\tau(a) \underset{D}{=} \begin{cases} 0 & \text{if } \sigma(a_0 \star \cdots \star a_{lg(a)-1}) = 0 \text{ and } \forall i < lg(a)\ \exists b \in \mathbb{N} \\ & [\, b \sqsubseteq a_0 \star \cdots \star a_i \wedge 0 < \gamma_i(b) \leq lg(a_0 \star \cdots \star a_i) + 1\,] \\ 1 & \text{else} \end{cases}$$

For α in τ put $i(\alpha) = \alpha(0) \star \alpha(1) \star \cdots$. Then i is a surjection from τ to σ, and it is easy to see that (τ, d) is steady •

THEOREM: the weakly stable closure of a spreadlike metric space is again spreadlike.

PROOF: combine the previous lemma with theorem 3.3.9 •

3.3.12 THEOREM: ($\mathbf{CP}_{\mathrm{ws}}$) let (σ,d) be a weakly stable metric spread. Let A be a subset of $X\times\mathbb{N}$ such that:

(i) $\forall\alpha\in\sigma\ \exists n\in\mathbb{N}\ [\,(\alpha,n)\in A\,]$

(ii) $\forall\alpha,\beta\in\sigma\ \forall n\in\mathbb{N}\ [\,(\alpha\equiv\beta\wedge(\alpha,n)\in A)\ \to\ (\beta,n)\in A\,]$

Then: $\forall\alpha\in\sigma\ \exists n,m\in\mathbb{N}\ \forall\beta\in\sigma\ [\,d(\alpha,\beta)<2^{-m}\ \to\ (\beta,n)\in A\,]$.

PROOF: by lemma 3.3.11, without loss of generality (σ,d) is steady. Let i be the canonical isometric embedding of (σ,d) into $(\Sigma_1(\sigma,d)\,,d)$, given by $i(\alpha)=\lessdot\alpha(0)+1,\ldots,\alpha(6)+1\gtrdot\star\lessdot\alpha(7)+1\gtrdot\star\cdots$. Since (σ,d) is weakly stable, i has an inverse j. We then have: $\forall\alpha\in\Sigma_1(\sigma,d)\ \exists n\in\mathbb{N}\ [\,(j(\alpha),n)\in A\,]$, so by the continuity principle $\mathbf{CP}$:

$$\forall\alpha\in\Sigma_1(\sigma,d)\ \exists n,s\in\mathbb{N}\ \forall\beta\in\Sigma_1(\sigma,d)\ [\,\overline{\alpha}(s)=\overline{\beta}(s)\ \to\ (j(\beta),n)\in A\,]$$

Now let α be in σ. We must produce $n,m\in\mathbb{N}$ such that for all $\beta\in\sigma$: $d(\alpha,\beta)<2^{-m}$ implies $(\beta,n)\in A$. Determine $n,s\in\mathbb{N}$, $s\geq 1$ such that for all γ in $\Sigma_1(\sigma,d)$: $\overline{\gamma}(s)=\overline{i(\alpha)}(s)$ implies $(j(\gamma),n)\in A$. Let β in σ such that $d(\alpha,\beta)<2^{-s-5}$. Then by definition of $\Sigma_1(\sigma,d)$ (see 3.3.9 and 3.3.8) there is a γ in $\Sigma_1(\sigma,d)$ such that $\overline{\gamma}(s)=\overline{i(\alpha)}(s)$ whereas $\gamma\equiv i(\beta)$. Therefore $(j(\gamma),n)\in A$, whereas $j(\gamma)\equiv\beta$. So by (ii) $(\beta,n)\in A$, meaning that we can take $m=s+5$ •

COROLLARY: every weakly stable spreadlike metric space is an apartness space.

PROOF: it suffices to prove that a weakly stable metric spread (σ,d) coincides identically with $(\sigma,\#_d)$. The only nontrivial implication is: if U is an open set in the $\#_d$-topology on σ, then U is open in (σ,d). Let β be in U. We have:

($\star$) $\forall\alpha\in\sigma\ \exists n\in\mathbb{N}\ [\,(n=0\wedge\alpha\#\beta)\vee(n=1\wedge\alpha\in U)\,]$

So put $A=\{(\alpha,n)\in\sigma\times\mathbb{N}\,|\,(n=0\wedge\alpha\#\beta)\vee(n=1\wedge\alpha\in U)\}$. Clearly A satisfies (i) and (ii) of our theorem above. Also: $(\beta,1)$ is in A, and $(\beta,0)$ is not in A. Applying the conclusion of the theorem to β we see: there is an $m\in\mathbb{N}$ such that $B(\beta,2^{-m})\subseteq U$ •

COROLLARY: let f be a weak function from a weakly stable spreadlike metric space (X,d) to another topological space $(Y,\mathcal{T})$. Then f is a function which in addition is continuous.

PROOF: without loss of generality X is a spread, say σ. We show that in fact f is a function. Let β,γ be in σ such that $f(\beta)\#f(\gamma)$. We have:

$(\star)$ $\forall\alpha\in\sigma\ \exists n\in\mathbb{N}\ [\,(n=0 \wedge f(\alpha)\#f(\beta)) \vee (n=1 \wedge f(\alpha)\#f(\gamma))\,]$

So put $A=\{\,(\alpha,n)\in\sigma\times\mathbb{N}\,|\,(n=0 \wedge f(\alpha)\#f(\beta)) \vee (n=1 \wedge f(\alpha)\#f(\gamma))\,\}$. Clearly A satisfies (i) and (ii) of our theorem above. Also: $(\beta,1)$ is in A, and $(\beta,0)$ is not in A. Applying the conclusion of the theorem to β we see: there is an $m\in\mathbb{N}$ such that for all α in $B(\beta,2^{-m})$: $f(\alpha)\#f(\gamma)$. Therefore $d(\beta,\gamma)\geq 2^{-m}$, so $\beta\#\gamma$, so f in fact is a function. But then f is continuous, by the previous corollary and theorem 1.1.0 •

3.3.13 LEMMA: let (A,d) be (strongly) traceable in a weakly stable spreadlike (X,d). Then (A,d) is (strongly) sublocated in (X,d).

PROOF: without loss of generality X is a spread, say σ. First let (A,d) be traceable in (σ,d). Let α be arbitrary in σ, and let m be arbitrary in $\mathbb{Z}$. It suffices to come up with an $a\in A$ such that $d(\alpha,a)<2^{-m}$ or with an $n\in\mathbb{N}$ such that $\forall a\in A\ [\,d(\alpha,a)>2^{-n}\,]$. To this end we define a subset B of $\sigma\times\mathbb{N}$ as follows:

$$B=\{(\beta,s)\mid(s=0 \wedge \exists a\in A\,[d(\beta,a)<2^{-m}]) \vee (s=1 \wedge \forall a\in A\,[\beta\#a])\}$$

Then for all β in X there is an $s\in\mathbb{N}$ such that $(\beta,s)\in B$, since (A,d) is traceable in (σ,d). Also, for $\beta,\gamma\in X$: if $(\beta,s)\in B$ and $\beta\equiv\gamma$, then $(\gamma,s)\in B$. So by $\mathbf{CP}_{\mathrm{ws}}$ (theorem 3.3.12) applied to α, we find $s,\ n\in\mathbb{N}$ such that $(\beta,s)\in B$ for all β in $B(\alpha,2^{-n})$. Now if $s=0$ then there is $a\in A$ such that $d(\alpha,a)<2^{-m}$. But if $s=1$, then for all β in $B(\alpha,2^{-n})$ and all a in A: $\beta\#a$. Then for all a in A: $d(\alpha,a)>2^{-n}$.

Now let (A,d) be strongly traceable in (σ,d). By lemma 1.3.3 this means that $(A,\#_d)$ is strongly sublocated in $(\sigma,\#_d)$. By $\mathbf{CP}_{\mathrm{ws}}$ (corollary 3.3.12) d metrizes the $\#_d$-topology on (σ,d), therefore (A,d) is strongly sublocated in (σ,d) •

3.3.14 another interesting aspect of $([0,1]_3,d_{\mathbb{R}})$ is its weakly stable closure $\overline{([0,1]_3,d_{\mathbb{R}})}$. We promised to show that $([0,1]_3,d_{\mathbb{R}})$ is NOT weakly stable (see 3.3.3) and also that $\overline{([0,1]_3,d_{\mathbb{R}})}$ is a sigma-compact apartness space which is NOT locally compact (see 2.3.2). These statements can be readily understood, but a precise proof is more difficult than it might seem at first glance. We will discuss how to build $\overline{([0,1]_3,d_{\mathbb{R}})}$ as a spread in a more convenient way than by proposition 3.3.8. Remember (3.3.6) that an alternative to the construction in 3.3.8 is to derive $W_1([0,1]_3,d_{\mathbb{R}})$ from the fan $[0,1]_3^{\mathbb{N}}$.

We make this precise. Let κ be the subfan of σ_ω determined by:

$\kappa = \{\alpha \in \sigma_\omega \mid \alpha(0) = 0 \wedge \forall n \in \mathbb{N}\,[\,\alpha_{[n]} \in [0,1]_3 \vee \alpha_{[n]} = \underline{3}\,]\,\}$

For α in κ put $\mathbb{N}_\alpha = \{n \in \mathbb{N} \mid \alpha_{[n]} \neq \underline{3}\}$. Notice that $\mathbb{N}_\alpha$ is a decidable subset of $\mathbb{N}$. Put $\kappa_0 = \{\alpha \in \kappa \mid \forall n \in \mathbb{N}\,[\alpha_{[n]} = \alpha_{[n+1]} \in [0,1]_3]\,\}$. Then κ_0 is the fan of all constant sequences $\alpha, \alpha, \alpha, \ldots$ in $[0,1]_3$. Define a function i from (κ_0, d_ω) to $([0,1], d_\mathbb{R})$ by putting: $i(\alpha) = \alpha_{[0]})$ for α in κ_0. Then $i(\alpha) = d_\mathbb{R}\text{-}\lim(\alpha_{[m]})_{m \in \mathbb{N}_\alpha}$ (this will shortly be generalized). Then obviously $(i(\kappa_0), d_\mathbb{R})$ coincides identically with $([0,1]_3, d_\mathbb{R})$, and so with $W_0([0,1]_3, d_\mathbb{R})$. Define a metric d_{ws} on κ_0 by: $d_{\mathrm{ws}}(\alpha,\beta) = d_\mathbb{R}(i(\alpha), i(\beta))$, for α, β in κ_1. We do not distinguish between $(\kappa_0, d_{\mathrm{ws}})$ and $(i(\kappa_0), d_\mathbb{R})$.

We turn to another subspread of κ which describes $W_1([0,1]_3, d_\mathbb{R})$. Put

$$\begin{aligned}\kappa_1 = \{\alpha \in \kappa \mid {} & \forall n \in \mathbb{N}\,[\,\alpha_{[n]} = \alpha_{[n+1]} \in [0,1]_3 \vee (\alpha_{[n+1]} = \underline{3} \wedge \overline{\alpha_{[n]}}(n) \approx_\mathbb{R} \overline{\alpha_{[n+2]}}(n))\,] \wedge \\ & \forall n, m \in \mathbb{N}\,[\,\alpha_{[n]} = \underline{3} = \alpha_{[m]} \rightarrow n = m\,]\,\}\end{aligned}$$

Then κ_1 is a fan which codes all sequences in $[0,1]_3$ which start out as a constant sequence $\beta, \beta, \beta, \ldots$ and allow for at most one 'jump' to another constant sequence $\gamma, \gamma, \gamma, \ldots$ which is close to β. In fact, if α is an element of κ_1, then such a jump is coded by an $n \in \mathbb{N}$ for which $\alpha_{[n]} = \underline{3}$. Then with β, γ as above we have that $\overline{\gamma}(n) \approx_\mathbb{R} \overline{\beta}(n)$.

Define a function i from (κ_1, d_ω) to $([0,1], d_\mathbb{R})$ by putting: $i(\alpha) = d_\mathbb{R}\text{-}\lim(\alpha_{[m]})_{m \in \mathbb{N}_\alpha}$ for α in κ_1. We leave it to the reader to verify that $(i(\kappa_1), d_\mathbb{R})$ coincides identically with $W_1([0,1]_3, d_\mathbb{R})$. Define a metric d_{ws} on κ_1 by: $d_{\mathrm{ws}}(\alpha,\beta) = d_\mathbb{R}(i(\alpha), i(\beta))$, for α, β in κ_1. We do not distinguish between $(\kappa_1, d_{\mathrm{ws}})$ and $(i(\kappa_1), d_\mathbb{R})$.

Finally, for $n \in \mathbb{N}$ we put:

$$\begin{aligned}\kappa_n = \{\alpha \in \kappa \mid {} & \forall s \in \mathbb{N}\,[\,\alpha_{[s]} = \alpha_{[s+1]} \in [0,1]_3 \vee (\alpha_{[s+1]} = \underline{3} \wedge \overline{\alpha_{[s]}}(s) \approx_\mathbb{R} \overline{\alpha_{[s+2]}}(s))\,] \wedge \\ & \forall m_0, \ldots, m_n \in \mathbb{N}\,[\,\forall i \leq n\,[\alpha_{[m_i]} = \underline{3}] \rightarrow \exists i < j \leq n\,[m_i = m_j]\,]\,\}\end{aligned}$$

Then κ_n is a fan which codes all sequences in $[0,1]_3$ which start out as a constant sequence $\beta, \beta, \beta, \ldots$ and allow for at most n 'jumps' to another constant sequence, which is sufficiently close to its 'predecessor'. Define a function i from (κ_1, d_ω) to $([0,1], d_\mathbb{R})$ by putting: $i(\alpha) = d_\mathbb{R}\text{-}\lim(\alpha_{[m]})_{m \in \mathbb{N}_\alpha}$ for α in κ_1. Define a metric d_{ws} on κ_n by: $d_{\mathrm{ws}}(\alpha,\beta) = d_\mathbb{R}(i(\alpha), i(\beta))$, for α, β in κ_n. We do not distinguish between $(\kappa_n, d_{\mathrm{ws}})$ and $(i(\kappa_n), d_\mathbb{R})$.

LEMMA: let $n \in \mathbb{N}$, then:

(i) $i(\kappa_n) \subseteq W_n([0,1]_3, d_\mathbb{R})$

(ii) $W_n([0,1]_3, d_{\mathbb{R}}) \subseteq i(\kappa_{2^n-1})$

PROOF: (i) is trivial. We prove (ii) by induction on $n \in \mathbb{N}$.
Basis: $n=0$. Trivially true.
Induction: let $n \in \mathbb{N}$ such that $W_n([0,1]_3, d_{\mathbb{R}}) \subseteq i(\kappa_{2^n-1})$. Let α be in $W_{n+1}([0,1]_3, d_{\mathbb{R}})$. Determine β in $W_n([0,1]_3, d_{\mathbb{R}})$ such that $\alpha \# \beta$ implies $\alpha \in W_n([0,1]_3, d_{\mathbb{R}})$. Determine β' in κ_{2^n-1} such that $\beta \equiv i(\beta')$. We describe an α' in $\kappa_{2^{n+1}-1}$ such that $\alpha \equiv i(\alpha')$. Let $m \in \mathbb{N}$. Suppose $\overline{\alpha}(2m) \approx \overline{\beta}(2m)$. Then put $\alpha'_{[m]} = \beta'_{[m]}$. Suppose $t = \mu s \in \mathbb{N}\,[\overline{\alpha}(2s) \not\approx \overline{\beta}(2s)]$. Then $\alpha \# \beta$, so $\alpha \in W_n([0,1]_3, d_{\mathbb{R}})$. So by the induction hypothesis we can determine δ in κ_{2^n-1} such that $\alpha \equiv i(\delta)$. Now put $\alpha'[t] = \underline{3}$ and for $m > t$ put $\alpha'[m] = \delta_{[m]}$. Clearly α' is in $\kappa_{2^{n+1}-1}$ and $\alpha \equiv i(\alpha')$ •

The lemma shows that $i(\bigcup_{n\in\mathbb{N}} \kappa_n) = \overline{\underline{([0,1]_3, d_{\mathbb{R}})}}$. It is therefore not necessary to distinguish between $(\bigcup_{n\in\mathbb{N}} \kappa_n, d_{\mathrm{ws}})$ and $\overline{([0,1]_3, d_{\mathbb{R}})}$. We can now prove that $([0,1]_3, d_{\mathbb{R}})$ is NOT weakly stable.

LEMMA: let a be in $\overline{\overline{\kappa}}_0$. Then: $\neg \forall \alpha \in \kappa_1 \cap a\ \exists \beta \in \kappa_0\ [\alpha \equiv_{\mathrm{ws}} \beta]$.

PROOF: we discuss the case $a = \lessdot\gtrdot$, the more general case is completely similar. Suppose:

$$(\star) \quad \forall \alpha \in \kappa_1\ \exists \beta \in \kappa_0\ [\alpha \equiv_{\mathrm{ws}} \beta].$$

Then by $\mathbf{AC}_{11}$ there is a spread-function γ from κ_1 to κ_0 realizing ($\star$). Let $\alpha_{\frac{1}{3},+}$ be the element of κ given by: $\forall n \in \mathbb{N}\,[\alpha_{\frac{1}{3},+[n]} = \lessdot 0,1 \gtrdot \star \underline{0}]$. Let $\alpha_{\frac{1}{3},-}$ be the element of κ given by: $\forall n \in \mathbb{N}\,[\alpha_{\frac{1}{3},-[n]} = \lessdot 0,0 \gtrdot \star \underline{2}]$. Then clearly: $\gamma(\alpha_{\frac{1}{3},+}) = \alpha_{\frac{1}{3},+}$ or $\gamma(\alpha_{\frac{1}{3},+}) = \alpha_{\frac{1}{3},-}$. We discuss the case $\gamma(\alpha_{\frac{1}{3},+}) = \alpha_{\frac{1}{3},+}$, the other case is similar. Determine $N \in \mathbb{N}$ such that for all β in κ_1: if $\overline{\beta}(N) = \overline{\alpha_{\frac{1}{3},+}}(N)$ then $\overline{\gamma(\beta)_{[0]}}(2) = \lessdot 0,1 \gtrdot$. Consider the sequence β in κ_1 given by: $\beta_{[i]} = \alpha_{\frac{1}{3},+}$ for $i \leq N$ and $\beta_{[N+1]} = \underline{3}$ and $\beta_{[m+N+2]} = \overline{\alpha}_{\frac{1}{3},-}(N) \star \underline{1}$ for $m \in \mathbb{N}$. Then clearly $\beta \not\equiv_{\mathrm{ws}} \gamma(\beta)$. Contradiction •

We will show by induction that for all $n \in \mathbb{N}$: $W_{n+1}([0,1]_3, d_{\mathbb{R}})$ does NOT coincide identically with $W_n([0,1]_3, d_{\mathbb{R}})$.

PROPOSITION: let $n \in \mathbb{N}$, and let a be in $\overline{\overline{\kappa}}_0$. Then: $\neg \forall \alpha \in \kappa_{n+1} \cap a\ \exists \beta \in \kappa_n\ [\alpha \equiv_{\mathrm{ws}} \beta]$.

PROOF: the proof of the proposition is by induction on $n \in \mathbb{N}$. The strategy is similar to the proof of the first lemma above.
Basis: $n=0$. This is just the first lemma above.

Induction: let $n\in\mathbb{N}$ be such that the lemma holds for n. We discuss the case $a=\langle\!\langle\,\rangle\!\rangle$, the general case is completely similar. Suppose:

$$(\star)\quad \forall\alpha\in\kappa_{n+2}\ \exists\beta\in\kappa_{n+1}\ [\,\alpha\equiv_{\mathrm{ws}}\beta\,].$$

Then by $\mathbf{AC}_{11}$ there is a spread-function γ from κ_{n+2} to κ_{n+1} realizing $(\star)$.

$\boxed{\text{claim}}$ $\gamma(\alpha_{\frac{1}{3},+})=\alpha_{\frac{1}{3},+}$ or $\gamma(\alpha_{\frac{1}{3},+})=\alpha_{\frac{1}{3},-}$.

$\boxed{\text{proof}}$ suppose that there is $m\in\mathbb{N}$ such that $\overline{\alpha_{\frac{1}{3},+}}(m)\neq\overline{\gamma(\alpha_{\frac{1}{3},+})}(m)\neq\overline{\alpha_{\frac{1}{3},+}}(m)$. Then since $\gamma(\alpha_{\frac{1}{3},+})\equiv_{\mathrm{ws}}\alpha_{\frac{1}{3},+}$ there must be an $s\in\mathbb{N}$ such that $\gamma(\alpha_{\frac{1}{3},+})_{[s]}=\underline{3}$. Determine $t\in\mathbb{N}$ such that for all α in κ_{n+2}: if $\overline{\alpha}(t)=\overline{\alpha_{\frac{1}{3},+}}(t)$, then $\gamma(\alpha)_{[s]}=\underline{3}$. Then clearly we find:

$$\forall\alpha\in\kappa_{n+2}\cap\overline{\alpha_{\frac{1}{3},+}}(t)\ \exists\beta\in\kappa_n\ [\,\alpha\equiv_{\mathrm{ws}}\beta\,]$$

But $\overline{\alpha_{\frac{1}{3},+}}(t)$ is in $\overline{\overline{\kappa}}_0$, so this contradicts the induction hypothesis $\circ$

We discuss the case $\gamma(\alpha_{\frac{1}{3},+})=\alpha_{\frac{1}{3},+}$, the other case is similar. Determine $N\in\mathbb{N}$ such that for all β in κ_1: if $\overline{\beta}(N)=\overline{\alpha_{\frac{1}{3},+}}(N)$ then $\overline{\gamma(\beta)_{[0]}}(2)=\langle\!\langle 0,1\rangle\!\rangle$. Consider the sequence β in κ_1 given by: $\beta_{[i]}=\alpha_{\frac{1}{3},+}$ for $i\leq N$ and $\beta_{[N+1]}=\underline{3}$ and $\beta_{[m+N+2]}=\overline{\alpha}_{\frac{1}{3},-}(N)\star\underline{1}$ for $m\in\mathbb{N}$. Then there must be a smallest $t\in\mathbb{N}$ such that $\gamma(\beta)_{[t]}=\underline{3}$ (since γ realizes $(\star)$). Then there must be a smallest $m\in\mathbb{N}$ such that $\overline{\gamma(\beta)}(m)\in\overline{\overline{\kappa}}_1$ and $\overline{\gamma(\beta)}(m)\notin\overline{\overline{\kappa}}_0$. Determine $s\in\mathbb{N}$ such that for all δ in κ_{n+2}: if $\overline{\delta}(s)=\overline{\beta}(s)$, then $\overline{\gamma(\delta)}(m)=\overline{\gamma(\beta)}(m)$. Then we see:

$$(\star\star)\quad \forall\delta\in\kappa_{n+2}\cap\overline{\beta}(s)\ \exists\eta\in\kappa_{n+1}\cap\overline{\gamma(\beta)}(m)\ [\,\delta\equiv_{\mathrm{ws}}\eta\,]$$

Let β' be the element of κ_0 determined by: $\beta'_{[p]}=\overline{\alpha}_{\frac{1}{3},-}(n)\star\underline{1}$ for all $p\in\mathbb{N}$. Then it is trivial to derive from $(\star\star)$:

$$\forall\delta\in\kappa_{n+1}\cap\overline{\beta'}(s)\ \exists\eta\in\kappa_n\ [\,\delta\equiv_{\mathrm{ws}}\eta\,]$$

This contradicts the induction hypothesis $\bullet$

COROLLARY:

(i) $\overset{\dots}{([0,1]_3,d_{\mathbb{R}})}$ is NOT compact.

(ii) $\overset{\dots}{([0,1]_3,d_{\mathbb{R}})}$ is NOT locally compact.

PROOF: for (i), suppose that $\overline{([0,1]_3, d_{\mathbb{R}})}$ is compact. Then $(\bigcup_{n\in\mathbb{N}}\kappa_n, d_{\mathrm{ws}})$ is compact, and therefore coincides with an apartness fan $(\tau, \#)$. Let h be a homeomorphism from $(\tau, \#)$ to $(\bigcup_{n\in\mathbb{N}}\kappa_n, d_{\mathrm{ws}})$. We find:

$$(\star) \qquad \forall\alpha\in\tau\ \exists n\in\mathbb{N}\ [\,h(\alpha)\in\kappa_n\,]$$

Then by the fan theorem **FT** there is an $N\in\mathbb{N}$ such that for all α in τ: $h(\alpha)\in\bigcup_{n\leq N}\kappa_n$. This contradicts the proposition. The argument for (ii) is similar, and left to the reader •

LEMMA: $\overline{([0,1]_3, d_{\mathbb{R}})}$ is a sigma-compact apartness space.

PROOF: clearly $\overline{([0,1]_3, d_{\mathbb{R}})}$ coincides with $(\bigcup_{n\in\mathbb{N}}\kappa_n, d_{\mathrm{ws}})$. Since $\overline{([0,1]_3, d_{\mathbb{R}})}$ is weakly stable, we have by the second corollary to theorem 3.3.12 that $(\bigcup_{n\in\mathbb{N}}\kappa_n, d_{\mathrm{ws}})$ coincides with $(\bigcup_{n\in\mathbb{N}}\kappa_n, \#)$, which is a sigma-compact space by definition •

This finishes our discussion of $\overline{([0,1]_3, d_{\mathbb{R}})}$.

CHAPTER FOUR

FUNCTIONAL TOPOLOGY

abstract

Using the material of chapter three, we prove the important Dugundji Extension Theorem, which holds already in Bishop's school. A consequence of the Dugundji theorem is that weakly stable convex subsets of a locally convex linear space are absolute retracts (**AR**'s). With the help of $\mathbf{AC}_{10}$ we go on to prove the Michael Selection Theorem. For important special cases the Dugundji Extension Theorem follows from the Michael theorem. Also a consequence of the Michael theorem is, that every continuous function from a spreadlike metric space to another metric space has a continuous modulus. Another application of the Michael theorem yields: if (A, d) is (strongly) traceable in a complete (X, d), then there is a strongly d-equivalent metric d' such that (A, d') is (strongly) halflocated in (X, d').

4.0 ABSOLUTE RETRACTS AND EXTENSORS

4.0.0* the fundamental situation in this chapter is that of a metric space (X,d), a subspace (A,d), and a continuous function f from (A,d) to (Y,d_Y), another metric space. The fundamental question that arises is: can we extend f to (X,d)?, that is: can we find a continuous $\tilde{f}$ from (X,d) to (Y,d_Y) such that the restriction of $\tilde{f}$ to (A,d) coincides with f?

Posed in full generality the fundamental question is unwieldy. Therefore we study spaces $(A,d),(X,d),(Y,d_Y)$ with special properties.

4.0.1* in the following let $(X,d),(A,d)$, f and (Y,d_Y) be as described in 4.0.0.

DEFINITION:

(i) a continuous function $\tilde{f}$ from (X,d) to (Y,d_Y) is called an *extension* of f to (X,d) (with respect to (Y,d_Y)) iff the restriction of $\tilde{f}$ to (A,d) coincides with f.

(ii) a continuous function $\tilde{f}$ from (X,d) to (A,d) is called a *retraction* of (X,d) onto (A,d) iff $\tilde{f}$ is an extension to (X,d) of id_A, the identity from (A,d) to (A,d).

REMARK: so $\tilde{f}\subseteq X\times Y$ is an extension of $f\subseteq A\times Y$ iff $f\subseteq\tilde{f}$.

4.0.2* DEFINITION:

(i) (A,d) is called a *retract* of (X,d) iff for all spaces (Y,d_Y) and all continuous f from (A,d) to (Y,d_Y), there is an extension $\tilde{f}$ of f to (X,d).

(ii) (Y,d_Y) is called an *extensor* of (X,d) iff for every strongly halflocated subspace (A,d) of (X,d) and every continuous f from (A,d) to (Y,d_Y), there is an extension $\tilde{f}$ of f to (X,d).

REMARK:

(1) (i) is easily seen equivalent to the more usual definition: (A,d) is a retract of (X,d) iff there is a retraction π of (X,d) onto (A,d).

(2) if (A,d) is a retract of (X,d), then (A,d) is closed in (X,d), and moreover: there is a d-equivalent metric d_π on X such that (A,d_π) is best approximable in (X,d_π) (see proposition 4.5.1).

(3) the condition in (ii) that (A,d) be strongly halflocated in (X,d), is justified a bit by the following considerations:

(I) suppose (A,d) is not closed in (X,d), or rather somewhat stronger: suppose there is a Cauchy-sequence $(a_n)_{n\in\mathbb{N}}$ in (A,d) such that $x = d\text{-}\lim(a_n)_{n\in\mathbb{N}}$ is in $X \backslash\backslash_A$, the strong complement of (A,d) in (X,d). Suppose moreover that there is an embedding of $(\mathbb{R}, d_{\mathbb{R}})$ in (Y, d_Y). Then there is a continuous f from (A,d) to (Y,d_Y) which cannot be extended to (X,d). (For convenience we postpone the proof of this statement until 4.1.2.)

(II) suppose no condition of locatedness is imposed at all. Let A be the set $\{0\} \cup \{1 \mid \exists n \in \mathbb{N}\ [n = k_{99}]\}$, and X, Y equal to $\{0,1\}$. Let $d = d_{\mathbb{R}} = d_Y$, and let f be the function from (A,d) to (Y,d_Y) defined by: $f(0)=0$ and $f(1) = [\frac{k_{99}+1}{2}] - [\frac{k_{99}}{2}]$. Then it is daring to say that f can be extended to (X,d).

(4) if (Y,d_Y) is an extensor of (X,d), and π is a retraction of (Y,d_Y) onto a subspace (B,d_Y), then (B,d_Y) is an extensor of (X,d). For let (A,d) be strongly halflocated in (X,d), and f a continuous function from (A,d) to (B,d_Y). Then since (Y,d_Y) is an extensor of (X,d), we can find a continuous $\tilde{f}$ from (X,d) to (Y,d_Y) which extends f, with respect to (Y,d_Y). But then $\pi \circ \tilde{f}$ extends f to (X,d) with respect to (B,d_Y).

4.0.3* DEFINITION:

(i) a space (Y,d_Y) is an *absolute retract* (**AR**) iff for every space (X,d), for every strongly halflocated subspace (A,d) of (X,d): if (Y,d_Y) is homeomorphic to (A,d), then (A,d) is a retract of (X,d).

(ii) a space (Y,d_Y) is an *absolute extensor* (**AE**) iff for every space (X,d), for every strongly halflocated subspace (A,d) of (X,d), and for every continuous f from (A,d) to (Y,d_Y), there is an extension of f to (X,d).

REMARK:

(1) (i) is easily seen to be equivalent with: (Y,d_Y) is an **AR** iff for every space (X,d), for every strongly halflocated subspace (A,d): every *homeomorphism* from (A,d) to (Y,d_Y) can be extended to (X,d). This shows that any **AE** is an **AR**.

(2) the first surprise is that nontrivial examples of **AR**'s and **AE**'s exist. The second is that every weakly stable **AR** is an **AE**.

(3) by remark 4.0.2 (4) we have: any retract of an **AE** is an **AE** itself.

The quantification over all spaces (Y, d_Y), all strongly halflocated subspaces (A, d), all continuous f from (A, d) to (X, d), etcetera, is a bit questionable. It might give the false impression that we have a good oversight of the collection of all metric spaces, and all continuous functions between them. Still we think that the definitions can be readily understood. Therefore we prefer to parallel certain classical definitions, see [vanMill89, sect.1.5]. In 4.2.6 we give an alternative characterization of weakly stable **AE**'s and weakly stable **AR**'s which is more down-to-earth.

4.1 the Dugundji Extension Theorem

4.1.0* in this section we will prove a fundamental result concerning the extension of a continuous function, the Dugundji Extension Theorem ([Dugundji51]). The proof is straightforward, but there is a slight technical problem which we prefer to treat with a definition and a separate lemma. The difficulty lies in that even for (A, d) strongly halflocated in (X, d), $(X \setminus\!\setminus_A, d)$ need not be separable, since we might not be able to indicate even one element of $X \setminus\!\setminus_A$. Of course the remedy is simple:

definition: let (X, d) be a metric space. Let ∞ be a formal element not contained in X, and u a fixed element in X. Let $X_\infty \underset{D}{=} X \cup \{\infty\}$. Define a metric $d_\infty = d_{\infty,u}$ on X_∞ by putting, for x, y in X: $d_\infty(x, y) \underset{D}{=} d(x, y)$, $d_\infty(x, \infty) \underset{D}{=} d(x, u) + 1$ and $d_\infty(\infty, \infty) \underset{D}{=} 0$. We often write (X_∞, d) for (X_∞, d_∞) (and d_∞ for $d_{\infty,u}$).

lemma: let (A, d) be strongly halflocated in (X, d) with parameter $D \in \mathbb{N}$, that is: $\forall x \in X\ \exists y \in A\ \forall a \in A\ [\, d(x, y) \leq D \cdot d(x, a)\,]$. Let $(x_s)_{s \in \mathbb{N}}$ be dense in (X, d). Then:

(i) there is a sequence $(a_s)_{s \in \mathbb{N}}$ in A such that, putting $\rho_s = d(x_s, a_s)$, we have: $\forall s \in \mathbb{N}\ \forall a \in A\ [\, \rho_s \leq D \cdot d(x_s, a)\,]$.

(ii) $(X \setminus\!\setminus_A \cup \{\infty\}, d_\infty)$ is separable.

(iii) putting $\rho_{-1} = 1$, $x_{-1} = \infty$ we have: $(B(x_s, \frac{\rho_s}{2D}))_{s \in \mathbb{N} \cup \{-1\}}$ is a per-enumerable cover of $(X \setminus\!\setminus_A \cup \{\infty\}, d_\infty)$.

(iv) there is a partition of unity $(p_s)_{s \in \mathbb{N}}$ subordinate to $(B(x_s, \frac{\rho_s}{2D}))_{s \in \mathbb{N}}$ on $(X \setminus\!\setminus_A, d)$ such that for all x in $X \setminus\!\setminus_A$: $p_s(x) > 0$ implies $\forall a \in A\ [\, d(x, a_s) < (2D+1) \cdot d(x, a)\,]$.

proof: for (i) notice that we have:

($\star$) $\forall s\in\mathbb{N}\ \exists y\in A\ \forall a\in A\ [\,d(x_s,y)\leq D\cdot d(x_s,a)\,]$.

by $\mathbf{AC}_{01}$ there is a function h from $\mathbb{N}$ to A realizing ($\star$). Now put $a_s=h(s)$ and $\rho_s=d(x_s,a_s)$. For (ii) we have:

($\star\star$) $\forall s,n\in\mathbb{N}\ \exists m\in\{0,1\}\ [\,(m=0\wedge 2^{-n}<\rho_s)\vee(m=1\wedge\rho_s<2^{-n+1})\,]$.

by $\mathbf{AC}_{00}$ there is a function h from $\mathbb{N}\times\mathbb{N}$ to $\{0,1\}$ realizing ($\star\star$). For $n\in\mathbb{N}$ define:

$$x_{s,n} \underset{D}{=} \begin{cases} \infty & \text{if } h(s,n)=0 \\ x_s & \text{if } h(s,n)=1 \end{cases}$$

Clearly the collection $(x_{s,n})_{s,n\in\mathbb{N}}\cup\{\infty\}$ is dense in $(X\setminus\setminus_A\cup\{\infty\},d_\infty)$. For (iii) put $x_{-1}=\infty$ and $\rho_{-1}=1$. Suppose $(x_{g(n)})_{n\in\mathbb{N}}$ d-converges to an x in $X\setminus\setminus_A\cup\{\infty\}$. Then obviously there is a δ in $\mathbb{R}^+$ such that for all $n\in\mathbb{N}$: $\frac{1}{2D}\cdot\rho_{g(n)}>\delta$. Now apply lemma 3.1.4. Lastly (iv): by theorem 3.1.2 there is a partition of unity $(q_n)_{n\in\mathbb{N}}$ subordinate to $(B(x_s,\frac{\rho_s}{2D}))_{s\in\mathbb{N}\cup\{-1\}}$. Then by lemma 3.1.2 there is a partition of unity $(p_s)_{s\in\mathbb{N}\cup\{-1\}}$ such that for all $s\in\mathbb{N}\cup\{-1\}$: $p_s^{-1}((0,1])\subseteq B(x_s,\frac{\rho_s}{2D})$. So $(p_s)_{s\in\mathbb{N}}$ is a partition of unity on $(X\setminus\setminus_A,d)$ subordinate to $(B(x_s,\frac{\rho_s}{2D}))_{s\in\mathbb{N}}$. Let x be in $X\setminus\setminus_A$ and suppose $p_s(x)>0$. Then $d(x,a_s)\leq d(x,x_s)+d(x_s,a_s)<\frac{2D+1}{2D}\cdot\rho_s$. But for all a in A we find, by our choice of a_s:

$$\left.\begin{array}{lll} d(x_s,a) & \geq & \frac{1}{D}\cdot\rho_s \\ d(x_s,x) & < & \frac{1}{2D}\cdot\rho_s \end{array}\right\}\text{ so } d(x,a)>\frac{1}{2D}\cdot\rho_s$$

Combined this gives $d(x,a_s)<(2D+1)\cdot d(x,a)$ for all a in A •

4.1.1* Theorem: (Dugundji Extension Theorem) let (A,d) be strongly halflocated in a metric space (X,d). Let f be a continuous function from (A,d) to a locally convex linear space (L,d_L). Then there is a continuous function $\tilde{f}$ from (X,d) to $W_1(conv(f(A)),d_L)$ such that f is the restriction of $\tilde{f}$ to (A,d).

proof: let $D\in\mathbb{N}_{\geq 1}$ such that $\forall x\in X\ \exists y\in A\ \forall a\in A\ [\,d(x,y)\leq D\cdot d(x,a)\,]$. Let $(x_s)_{s\in\mathbb{N}}$ be dense in (X,d), and let $(a_s)_{s\in\mathbb{N}}$ and $(p_s)_{s\in\mathbb{N}}$ be as in lemma 4.1.0 (i) and (iv) respectively.

$\boxed{\text{claim}}$ $\forall x\in X\quad \exists^{!\equiv}z\in W_1(conv(f(A))\quad [\,(x\in A\ \rightarrow\ z\equiv f(x))\ \wedge\ (x\in X\setminus\setminus_A\ \rightarrow\ z\equiv\sum_s p_s(x)\cdot f(a_s)\,]$

$\boxed{\text{proof}}$ let $x\in X$, find $x_A\in A$ such that $\forall a\in A\ [\,d(x,x_A)\leq D\cdot d(x,a)\,]$. Since (L,d_L) is locally convex, for each $n\in\mathbb{N}$ there is a convex open neighbourhood U_n of $f(x_A)$ such that d_L-diam$(U_n) < 2^{-n}$. Determine a sequence $(\delta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ such that for all $n\in\mathbb{N}$ we have $\delta_{n+1} < \frac{1}{2}\delta_n$ and: $\forall a\in A\ [\,d(a,x_A)<\delta_n \rightarrow f(a)\in U_n\,]$. As usual we obtain:

$$(\star)\quad \forall n\in\mathbb{N}\ \exists (m,y)\in\{0,1\}\times conv(f(A))\ [\,(m=0 \wedge d(x,x_A)<\tfrac{\delta_n}{2D+2} \wedge y=f(x_A)) \vee (m=1 \wedge d(x,x_A)>\tfrac{\delta_n}{4D+4} \wedge y=\textstyle\sum_s p_s(x)\cdot f(a_s))\,]$$

By $\mathbf{AC}_{01}$ there is a function h from $\mathbb{N}$ to $\{0,1\}\times conv(f(A))$ realizing $(\star)$. Let h_0 and h_1 be functions from $\mathbb{N}$ to $\{0,1\}$ and $conv(f(A))$ respectively, such that for all $n\in\mathbb{N}$: $h(n)=(h_0(n),h_1(n))$. Put $z_n=h_1(n)\in conv(f(A))$, for $n\in\mathbb{N}$. We will see that $(z_n)_{n\in\mathbb{N}}$ is d_L-Cauchy, by showing that for all $n\in\mathbb{N}$ we can decide: $d(z_{n+1},f(x_A))<2^{-n}$ or $\forall m\geq n\,[z_m=z_{n+1}]\,]$. For this let $n\in\mathbb{N}$.

$\boxed{\text{case 1}}$ $h_0(n+1)=0$
Then $z_{n+1}=f(x_A)$ and there is little to prove.

$\boxed{\text{case 2}}$ $h_0(n+1)=1$
Then $z_{n+1}=\sum_s p_s(x)\cdot f(a_s)$.

$\boxed{\text{case 2.1}}$ $h_0(n)=1$
Then $d(x,x_A)>\frac{\delta_n}{4D+4}$. Since for all $m\in\mathbb{N}$: $\delta_{m+1}<\frac{1}{2}\delta_m$, we must have $\forall m\geq n\ [\,z_m=\sum_s p_s(x)\cdot f(a_s)\,]$

$\boxed{\text{case 2.2}}$ $h_0(n)=0$
But then $d(x,x_A)<\frac{\delta_n}{2D+2}$, so by lemma 4.1.0 (iv) $d(x,a_s)<\frac{2D+1}{2D+2}\cdot\delta_n$ for all $s\in\mathbb{N}$ such that $p_s(x)>0$. Therefore $d(x_A,a_s)<\delta_n$ for all $s\in\mathbb{N}$ such that $p_s(x)>0$. And so $z_{n+1}=\sum_s p_s(x)\cdot f(a_s)$ is a convex combination of elements of U_n, therefore in U_n itself. Observe that $f(x_A)\in U_n$ and d_L-diam$(U_n) < 2^{-n}$.

Put $z=d_L$-lim$(z_n)_{n\in\mathbb{N}}$. Now $z\,\#\,f(x_A)$ implies $z\in conv(f(A))$. So $z\in W_1(conv(f(A),d_L)$. Clearly z satisfies our claim, but we still have to show uniqueness. This is easy: suppose $v\in W_1(conv(f(A),d_L)$ such that $x\in A$ implies $v\equiv f(x)$ and $x\in X\setminus\setminus_A$ implies $v\equiv\sum_s p_s(x)\cdot f(a_s)$. Suppose $z\,\#\,v$. Then clearly $x\notin A$ and also $x\notin X\setminus\setminus_A$. But $x\notin X\setminus\setminus_A$ implies that $d(x,x_A)\equiv 0$, which implies that $x\in A$. Contradiction, therefore $z\equiv v$ $\circ$

By the claim we may define a function $\tilde{f}$ from X to $W_1(conv(f(A),d_L)$, putting $\tilde{f}(x)=z$ with z as in the claim.

$\boxed{\text{claim}}$ $\tilde{f}$ is a continuous function from (X,d) to $W_1(conv(f(A),d_L)$.

$\boxed{\text{proof}}$ let $x \in X$, find $x_A \in A$ such that for all $a \in A$: $d(x, x_A) \leq D \cdot d(x, a)$. Determine a sequence $(\delta_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^+$ such that $\delta_{n+1} < \frac{1}{2}\delta_n$ and: $\forall a \in A\ [d(a, x_A) < \delta_n \rightarrow f(a) \in U_n]$. Let n be arbitrary in $\mathbb{N}$. We will show that there is an ϵ in $\mathbb{R}^+$ such that for all y in X: $d(x, y) < \epsilon$ implies $d_L(\tilde{f}(x), \tilde{f}(y)) \leq 2^{-n+1}$.

For all a in A: $d(a, x_A) < \delta_n$ implies $d_L(f(a), f(x_A)) < 2^{-n}$. Also, by lemma 4.1.0 (iv), we have: $\forall s \in \mathbb{N}\ [p_s(x) > 0 \rightarrow d(x, a_s) \leq (2D+1) \cdot d(x, x_A)]$. Therefore: $\forall s \in \mathbb{N}\ [p_s(x) > 0 \rightarrow d(a_s, x_A) \leq (2D+2) \cdot d(x, x_A)]$. Here we go again, letting $y \in X$:

$\boxed{\text{case 1}}$ $d(x, x_A) < \frac{\delta_n}{2D+2}$

Then $d(y, x) < \frac{\delta_n}{2D+2} - d(x, x_A)$ implies $d(y, x_A) < \frac{\delta_n}{2D+2}$. By lemma 4.1.0 (iv) we find $\forall s \in \mathbb{N}\ [p_s(y) > 0 \rightarrow d(y, x_A) \leq (2D+1) \cdot d(y, x_A)]$, and so $\forall s \in \mathbb{N}\ [p_s(y) > 0 \rightarrow d(a_s, x_A) \leq \delta_n]$. Clearly both $y \in A$ and $y \in X \setminus\!\setminus_A$ imply $d_L(f(y), f(x_A)) < 2^{-n}$. Therefore $d_L(f(y), f(x_A)) \leq 2^{-n}$. So we also have: $d_L(f(x), f(x_A)) \leq 2^{-n}$. Combined this gives $d_L(f(y), f(x)) \leq 2^{-n+1}$. So we can take $\epsilon = \frac{\delta_n}{2D+2} - d(x, x_A)$.

$\boxed{\text{case 2}}$ $d(x, x_A) > \frac{\delta_n}{4D+4}$

Then $x \in X \setminus\!\setminus_A$, and so $d(y, x) < \frac{\delta_n}{D(4D+4)}$ implies $y \in X \setminus\!\setminus_A$ which in turn implies $\tilde{f}(y) = \sum_s p_s(y) \cdot f(a_s)$. Clearly this is a continuous expression on $(X \setminus\!\setminus_A, d)$, so we can find an ϵ in $\mathbb{R}^+$, $\epsilon < \frac{\delta_n}{D(4D+4)}$ such that $d(y, x) < \epsilon$ implies $d_L(f(y), f(x)) \leq 2^{-n+1}$ $\circ$

Verifying that f is the restriction of $\tilde{f}$ to (A, d) is trivial $\bullet$

COROLLARY: let (B, d_L) be a weakly stable convex subspace of a locally convex linear space (L, d_L). Then (B, d_L) is an **AE**.

REMARK: if (X, d) is spreadlike, then we can weaken the condition that (A, d) be strongly halflocated in (X, d) to the condition that (A, d) be strongly traceable in (X, d), see theorem 4.5.5.

4.1.2* we can now prove remark 4.0.2 (3)(i). We copy the notations from there. Since x is in $X \setminus\!\setminus_A$, without loss of generality we may assume that for $n, m \in \mathbb{N}$: $n \neq m$ implies $a_n \# a_m$.

$\boxed{\text{claim}}$ $(\{a_n \mid n \in \mathbb{N}\}, d)$ is strongly halflocated in (A, d).

$\boxed{\text{proof}}$ this is an easy consequence of the following observation: if a is in A, then we can calculate $\delta = d(a, x) \in \mathbb{R}^+$. Since $(a_n)_{n \in \mathbb{N}}$ d-converges to x, there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\frac{2}{3}\delta < d(a,a_n) < \frac{4}{3}\delta$. Of course since $n \neq m$ implies $a_n \# a_m$, we can either find an $m < N$ such that $d(a,a_m) < \frac{2}{3}\delta$ and for all $n \in \mathbb{N}$: $d(a,a_m) \leq 3 \cdot d(a,a_n)$, or we find: for all $m < N$: $d(a,a_m) > \frac{1}{2}\delta$, and then for all $n \in \mathbb{N}$: $d(a,a_N) \leq 3 \cdot d(a,a_n)$ ∘

Next define a continuous g from $(\{a_n | n \in \mathbb{N}\}, d)$ to $(\mathbb{R}, d_{\mathbb{R}})$ by putting $g(a_{2n}) = 1$ and $g(a_{2n+1}) = -1$. By the Dugundji Extension Theorem 4.1.1 there is an extension $\tilde{g}$ of g to (A,d) . Let i be an embedding of $(\mathbb{R}, d_{\mathbb{R}})$ in (Y, d_Y) . Clearly $f = i \circ \tilde{g}$ is a continuous function from (A,d) to (Y, d_Y) which cannot be extended to (X,d) .

4.2 A NORMED LINEAR ISOMETRICAL EXTENSION OF (X,d)

4.2.0* normed linear spaces were defined in chapter zero. Our primary aim in this section is to show that each metric space (X,d) coincides isometrically with a halflocated subspace of a normed linear space (X^*, d^*) . Moreover (X, d^*) is strongly halflocated in (X^*, d^*) if (X,d) is weakly stable. These results, very interesting in their own right, we depend on not only to show that a weakly stable **AR** is an **AE**, but also to prove the fundamental theorems 4.5.2, 4.5.3, and 4.5.4 which connect 'topologically halflocated in a metric spread' to 'sublocated in a metric spread'.

Some of these results can also be obtained in the following way. Let (X,d) be a metric space. By lemma 2.4.7 there is an embedding j of (X,d) in the Hilbert cube $(\mathcal{Q}, d_{\mathcal{Q}})$. Write $C_{\mathcal{Q},\mathbb{R},\| \|_{\sup}}$ for the space $(C((\mathcal{Q}, d_{\mathcal{Q}}), (\mathbb{R}, d_{\mathbb{R}})), \| \|_{\sup})$ of all continuous spread-functions from $(\mathcal{Q}, d_{\mathcal{Q}})$ to $(\mathbb{R}, d_{\mathbb{R}})$, endowed with the supremum norm (see 0.5.3). By corollary 0.5.6 $C_{\mathcal{Q},\mathbb{R},\| \|_{\sup}}$ is a Banach space. It is not difficult to see that the function i from $(\mathcal{Q}, d_{\mathcal{Q}})$ to $C_{\mathcal{Q},\mathbb{R},\| \|_{\sup}}$ defined by $i(x)(y) \underset{D}{=} d_{\mathcal{Q}}(x,y)$ is an isometrical embedding of $(\mathcal{Q}, d_{\mathcal{Q}})$ in $C_{\mathcal{Q},\mathbb{R},\| \|_{\sup}}$. By a similar reasoning as the one put forward in this section, we can show that $(i \circ j(X), d_{\| \|})$ is halflocated in $(\mathit{conv}\,(i \circ j(X)), d_{\| \|})$, and strongly halflocated in $(\mathit{conv}\,(i \circ j(X)), d_{\| \|})$ if (X,d) is weakly stable. This suffices to prove theorem 4.2.5 and theorem 4.5.2. However, this development has one drawback which we think serious enough: the resulting $d_{\| \|}$ restricted to X in general is not strongly d-equivalent, let alone isometric to d . So $\overline{(X,d)}$ then in general is not homeomorphic to $\overline{(i \circ j(X), d_{\| \|})}$. If e.g. we start out with a complete non-compact space, we lose the completeness in the process. Therefore we would be unable to prove all of theorem 4.5.3.

Defining the normed linear space is not too difficult, but proving the definition correct will cost us more than just a lemma. First we need a preliminary definition.

DEFINITION: let x be in $\mathbb{R}$. Then we write x^+ for $\inf(\sup(0,x),|x|)$ and x^- for $\inf(\sup(0,-x),|x|)\equiv(-x)^+$.

So then for all x in $\mathbb{R}$: $x^+, x^-\in\mathbb{R}_{\geq 0}$ and $x\equiv x^+ - x^-$. We are ready to define our normed linear space. Please remember the definition of (X_∞, d) in 4.1.0.

DEFINITION: let (X,d) be a space. Then $X^* \underset{D}{=} \bigcup_{n\in\mathbb{N}} \mathbb{R}^n\times(X_\infty)^n$. We define a function $\oplus$ from $X^*\times X^*$ to X^* as follows. Let $x=(\rho_0,\ldots,\rho_n,x_0,\ldots,x_n)$ and $y=(\sigma_0,\ldots,\sigma_m,y_0,\ldots,y_m)$ be in X^*. Then:

$x\oplus y \underset{D}{=} (\rho_0,\ldots,\rho_n,\sigma_0,\ldots,\sigma_m,x_0,\ldots,x_n,y_0,\ldots,y_m)$.

We also define a function $\odot$ from $\mathbb{R}\times X^*$ to X^*. Let σ be in $\mathbb{R}$, and $x=(\rho_0,\ldots,\rho_n,x_0,\ldots,x_n)$ in X^*. Then:

$\sigma\odot x \underset{D}{=} (\sigma\rho_0,\ldots,\sigma\rho_n,x_0,\ldots,x_n)$.

For an element $(\rho_0,\ldots,\rho_n,x_0,\ldots,x_n)$ of X^* we write $\bigoplus_{i\leq n}\rho_i\cdot x_i$, more simply $\rho_0\cdot x_0$ if $n=0$, and x_0 for $1\cdot x_0$. We write $\ominus x$ for $-1\odot x$ and $x\ominus y$ for $x\oplus(\ominus y)$. Finally, we write x^+ for $\bigoplus_{i\leq n}\rho_i^+\cdot x_i$ and x^- for $\bigoplus_{i\leq n}\rho_i^-\cdot x_i$. Put $\mathbf{0}\underset{D}{=}\infty$. Define:

$X^*_{\mathbb{Q},\geq 0} \underset{D}{=} \{\bigoplus_{i\leq n} p_i\cdot x_i \mid (x_i)_{i\leq n}\in X_\infty\ (p_i)_{i\leq n}\in\mathbb{Q}_{\geq 0} \mid n\in\mathbb{N}\}$,

$X^*_{\mathbb{R},\geq 0} \underset{D}{=} \{\bigoplus_{i\leq n} \rho_i\cdot x_i \mid (x_i)_{i\leq n}\in X_\infty\ (\rho_i)_{i\leq n}\in\mathbb{R}_{\geq 0} \mid n\in\mathbb{N}\}$, and

$X^*_\alpha \underset{D}{=} \{\bigoplus_{i\leq n}\rho_i\cdot x_i\in X^*_{\mathbb{R},\geq 0} \mid \sum_i\rho_i\equiv\alpha\}$ for $\alpha\in\mathbb{R}_{\geq 0}$.

First, for each α in $\mathbb{R}_{\geq 0}$, we define a metric d_{eq} on X^*_α as follows. Let $\bigoplus_{i\leq n}\rho_i\cdot x_i$ and $\bigoplus_{j\leq m}\sigma_j\cdot y_j$ be in X^*_α. Then:

$d_{eq}(x,y) \underset{D}{=} \inf(\{\sum_{i,j}\tau_{i,j}\cdot d(x_i,y_j) \mid \tau_{i,j}\in\mathbb{R}_{\geq 0} \mid \sum_j\tau_{i,j}\equiv\rho_i,\ \sum_i\tau_{i,j}\equiv\sigma_j \mid i\leq n,\ j\leq m\})$

We define a metric $d_{\geq 0}$ on $X^*_{\mathbb{R},\geq 0}$. Let $x=\bigoplus_{i\leq n}\rho_i\cdot x_i$ and $y=\bigoplus_{j\leq m}\sigma_j\cdot y_j$ be in $X^*_{\mathbb{R},\geq 0}$, with $\sum_i\rho_i=P$, $\sum_j\sigma_j=Q$. Then $d_{\geq 0}(x,y)\underset{D}{=} d_{eq}(x\oplus Q\cdot\infty, y\oplus P\cdot\infty)$. Now we define a metric d^* on all of X^*. Let x,y be in X^*. Then $d^*(x,y)\underset{D}{=} d_{\geq 0}(x^+\oplus y^-, y^+\oplus x^-)$.

Finally, define a function $\|\ \|^*$ from X^* to $\mathbb{R}_{\geq 0}$, by putting $\|x\|^*\underset{D}{=} d^*(x,\mathbf{0})$.

REMARK: the reader might benefit from the guiding idea behind this definition: think of the elements of X simultaneously as chemical compounds and as depots for chemical compounds. The chemical compounds have quantity 1, the depots have capacity 1. The cost of storing x in depot y is equal to $d(x,y)$. Of course, multiplying both quantity and capacity by $\rho\in\mathbb{R}_{\geq 0}$ results in a proportional cost of $\rho\cdot d(x,y)$. Now if $x=\bigoplus_{i\leq n}\rho_i\cdot x_i$ and $y=\bigoplus_{j\leq m}\sigma_j\cdot y_j$ are in X^*_α, then we wish to store a quantity ρ_i of each x_i in a collection of depots y_j, each with (temporary) capacity σ_j. We can precisely store the total quantity of $\sum_i\rho_i=\sum_j\sigma_j=\alpha$ and for this quantity we seek to minimize the cost. If more generally x,y are in $X^*_{\mathbb{R},\geq 0}$, then to avoid leftovers we balance the quantity of x and the capacity of y with the 'neutral' compound/depot ∞. If most generally x and y are in X^*, then we interpret negative coefficients in the obvious way. Now $d^*(x,y)$ is just the total cost in this operation.

With the next series of lemmas we hope to achieve peace of mind about our definition. We show that d_{eq} can actually be calculated, and that it satisfies the triangle inequality, and therefore is a metric (obviously $d_{eq}(x,y)\equiv d_{eq}(y,x)\geq 0$ and $d_{eq}(x,x)\equiv 0$). The same then is easily seen to hold for d^*, by proving $\|\ \|^*$ a norm and d^* to coincide with $d_{\|\ \|^*}$. By this time it will be obvious that $\oplus$ respects the d^*-equivalence, etcetera.

4.2.1* to see that d_{eq} is actually calculable, therefore well-defined, let $x=\bigoplus_{i\leq n}\rho_i\cdot x_i$ and , $y=\bigoplus_{j\leq m}\sigma_j\cdot y_j$ be in X^*_α. Define $f:\mathbb{R}^{nm}\to\mathbb{R}_{\geq 0}$ by $f((\tau_{i,j})_{i,j\leq n,m})\underset{D}{=}\sum_{i,j}\tau_{i,j}\cdot d^*(x_i,y_j)$. Then clearly f is uniformly continuous. Notice that the set $\{\,(\tau_{i,j})_{i,j\leq n,m}\in\mathbb{R}^{nm}_{\geq 0}\ \mid\ \sum_j\tau_{i,j}\equiv\rho_i,\ \sum_i\tau_{i,j}\equiv\sigma_j\,\}$ is precompact, and that the function to be minimized on this set (in order to calculate $d_{eq}(x,y)$) is f. Now use lemma 0.4.3.

REMARK: also notice that the uniform continuity of f implies the following: if for $i\leq n,\ j\leq m\ :\ \rho_i=d_{\mathbb{R}}\text{-}\lim(p_{i,s})_{s\in\mathbb{N}},\ \sigma_j=d_{\mathbb{R}}\text{-}\lim(q_{j,s})_{s\in\mathbb{N}}$ such that moreover for all $s\in\mathbb{N}\ :\ p_{i,s},\,q_{j,s}\in\mathbb{Q}_{\geq 0}$ and $\sum_i p_{i,s}=\sum_j q_{j,s}$, then $d_{eq}(x,y)=d_{\mathbb{R}}\text{-}\lim(d_{eq}(x_s,y_s))_{s\in\mathbb{N}}$, where $x_s=\bigoplus_{i\leq n}p_{i,s}\cdot x_i,\ y_s=\bigoplus_{j\leq m}q_{j,s}\cdot y_j$.

DEFINITION: let $x=\bigoplus_{i\leq n}\rho_i\cdot x_i$ and $y=\bigoplus_{j\leq m}\sigma_j\cdot y_j$ be in X^*_α.

(i) an element $(\tau_{i,j})_{i,j\leq n,m}$ of $\mathbb{R}^{nm}_{\geq 0}$ such that $\sum_j\tau_{i,j}\equiv\rho_i,\ \sum_i\tau_{i,j}\equiv\sigma_j$, is called a *distribution* of x in y.

(ii) a distribution $(\tau_{i,j})_{i,j\leq n,m}$ of x in y is called *rational* iff $(\tau_{i,j})_{i,j\leq n,m}$ is in $\mathbb{Q}^{nm}_{\geq 0}$.

LEMMA: let $x=\bigoplus_{i\leq n}\rho_i\cdot x_i$, $y=\bigoplus_{j\leq m}\sigma_j\cdot y_j$, $z=\bigoplus_{k\leq s}\tau_k\cdot z_k$ be elements of $X^*_{\mathbb{R},\geq 0}$, such that $P=\sum_i\rho_i=\sum_j\sigma_j$. Put $R=\sum_k\tau_k$, then

(i) $d_{eq}(x\oplus(0\odot z),y)\equiv d_{eq}(x,y)$, $d_{eq}(x\oplus z,y\oplus R\cdot\infty)\equiv d_{eq}(y\oplus R\cdot\infty,z\oplus x)$.

(ii) $d_{eq}(x\oplus z,y\oplus z)\equiv d_{eq}(x,y)$

(iii) if moreover $R \equiv P$, then $d_{eq}(x,z) \le d_{eq}(x,y) + d_{eq}(y,z)$.

PROOF: (i) is a triviality. For the rest of the lemma, please reread remark 4.2.0. First let $x = \bigoplus_{i\le n} p_i \cdot x_i$, $y = \bigoplus_{j\le m} q_j \cdot y_j$, $z = \bigoplus_{k\le s} r_k \cdot z_k$ be in $X^*_{\mathbb{Q},\ge 0}$. We certainly have: $d_{eq}(x \oplus z, y \oplus z) \le d_{eq}(x,y)$, since we can always extend a distribution of x in y to a distribution of $x \oplus z$ in $y \oplus z$ at zero cost, by putting z in z in the obvious way. For the reverse equation, we consider a rational distribution $(t_{u,v})_{u,v\le n+s+1, m+s+1}$ of $x \oplus z$ in $y \oplus z$ and distinguish two cases.

$\boxed{\text{case 1}}$ for $j \le m$, $k \le s$ we have $t_{n+k+1,j} = 0$
this means that all of z gets put into z, and clearly the cost of this distribution is at least $d(x,y)$.

$\boxed{\text{case 2}}$ there are $j \le m$, $k \le s$ such that $t_{n+k+1,j} > 0$
this means that some of z gets put into y, specifically: $t_{n+k+1,j}$ of z_k is put into depot $q_j \cdot y_j$. Then at least $t_{n+k+1,j}$ of depot $r_k \cdot z_k$ is filled up with other compounds than z_k, say $w_0, .., w_l \in \{x_0, .., x_n, z_0, .., z_{k-1}, z_{k+1}, .., z_s\}$. So there are $u_0, .., u_l \ne n+k+1$ such that $\sum_{e\le l} t_{u_e, m+k+1} \ge t_{n+k+1,j}$. Clearly we cannot lose if we interchange the amount $t_{n+k+1,j}$ of z_k in $q_j \cdot y_j$ with the same amount of the w_e's in depot $r_k \cdot z_k$, since for all $e \le l$: $d(w_e, z_k) + d(z_k, y_j) \ge d(w_e, y_j)$.

Iterating the above argument, we see that we can better our distribution in such a way that case 2 is eliminated altogether. By the continuity of f (see our remark above), and since any distribution is a limit of rational distributions, we find $d_{eq}(x,y) \le d_{eq}(x \oplus z, y \oplus z)$.

Then (iii). We must show that it is never cheaper to first put x in y and then empty y into z, than to put x directly into z. Let $(\rho_{i,j})_{i,j\le n,m}$ and $(\sigma_{j,k})_{j,k\le m,s}$ be in $\mathbb{R}_{\ge 0}$ such that:

$$\begin{array}{ll} \sum_i \rho_{i,j} \equiv q_j & \sum_j \sigma_{j,k} \equiv r_k \\ \sum_j \rho_{i,j} \equiv p_i & \sum_k \sigma_{j,k} \equiv q_j \end{array}$$

Put $\tau_{i,k} = \sum_j \rho_{i,j} \cdot \frac{\sigma_{j,k}}{q_j}$ (if $q_j = 0$, which is decidable, then with '$\frac{\sigma_{j,k}}{q_j}$' we also mean 0; in this case $\rho_{i,j}$ and $\sigma_{j,k}$ are also 0 for all $i \le n$ and $k \le s$). Then $\sum_i \tau_{i,k} \equiv \sum_j (\frac{\sigma_{j,k}}{q_j} \cdot \sum_i \rho_{i,j}) \equiv \sum_j \sigma_{j,k} \equiv r_k$ and $\sum_j \tau_{i,k} \equiv \sum_j (\rho_{i,j} \cdot \sum_k \frac{\sigma_{j,k}}{q_j}) \equiv \sum_j \rho_{i,j} \equiv p_i$, so $(\tau_{i,k})_{i,k\le n,s}$ is a distribution of x in z. (In fact $\tau_{i,k}$ is the amount of x_i which ends up in z_k, if we follow the distributions $(\rho_{i,j})_{i,j\le n,m}$ and $(\sigma_{j,k})_{j,k\le m,s}$ linearly). Now by the triangle inequality for d on X_∞ we have:

$$\tau_{i,k} \cdot d(x_i, z_k) \le \sum_j \rho_{i,j} \cdot \frac{\sigma_{j,k}}{q_j} \cdot (d(x_i, y_j) + d(y_j, z_k))$$

$\sum_{i,k} \tau_{i,k} \cdot d(x_i, z_k) \leq \sum_{i,j}(\rho_{i,j} \cdot \sum_k \frac{\sigma_{j,k}}{q_j} \cdot d(x_i, y_j)) + \sum_{j,k}(\frac{\sigma_{j,k}}{q_j} \cdot \sum_i \rho_{i,j} \cdot d(y_j, z_k)))$

$\sum_{i,k} \tau_{i,k} \cdot d(x_i, z_k) \leq \sum_{i,j} \rho_{i,j} \cdot d(x_i, y_j) + \sum_{j,k} \sigma_{j,k} \cdot d(y_j, z_k)$

which implies $d_{eq}(x,z) \leq d_{eq}(x,y) + d_{eq}(y,z)$ by the arbitrariness of $(\rho_{i,j})_{i,j\leq n,m}$ and $(\sigma_{j,k})_{j,k\leq m,s}$.

We have thus proved the lemma for x, y, z in $X^*_{\mathbb{Q},\geq 0}$. Now for x, y, z not necessarily in $X^*_{\mathbb{Q},\geq 0}$, we can simply take limits, by our remark above •

COROLLARY: let x, y, z be in $X^*_{\mathbb{R},\geq 0}$. Then:

(1) $d_{\geq 0}(x \oplus (0 \odot z), y) \equiv d_{\geq 0}(x,y)$ and $d_{\geq 0}(x \oplus z, y \oplus R \cdot \infty) \equiv d_{\geq 0}(y, z \oplus x)$.

(2) $d_{\geq 0}(x \oplus z, y \oplus z) \equiv d_{\geq 0}(x,y)$

(3) $d_{\geq 0}(x,z) \leq d_{\geq 0}(x,y) + d_{\geq 0}(y,z)$.

PROOF: (1) and (2) follow directly from (i), (ii), and the definition of $d_{\geq 0}$. Now (ii) implies that $d_{\geq 0}(x,z) \equiv d_{eq}(x \oplus (Q+R) \cdot \infty, z \oplus (Q+P) \cdot \infty)$ and $d_{\geq 0}(x,y) \equiv d_{eq}(x \oplus (Q+R) \cdot \infty, y \oplus (R+P) \cdot \infty)$ and $d_{\geq 0}(y,z) \equiv d_{eq}(y \oplus (R+P) \cdot \infty, z \oplus (Q+P) \cdot \infty)$, implying (3) by (iii) •

REMARK: this lemma shows that in fact $d_{\geq 0}$ coincides with d_{eq} on X^*_α. But also d^* coincides with $d_{\geq 0}$ on $X^*_{\mathbb{R},\geq 0}$. This follows immediately from (1) and (2).

4.2.2* LEMMA: let x, y, z, w be elements of $X^*_{\mathbb{R},\geq 0}$. Then $d_{\geq 0}(x \oplus y, z \oplus w) \leq d_{\geq 0}(x,z) + d_{\geq 0}(y,w)$

PROOF: by corollary 4.2.1 (1) and (2) we find: $d_{\geq 0}(x,z) \equiv d_{\geq 0}(x \oplus w, z \oplus w)$ and $d_{\geq 0}(y,w) \equiv d_{\geq 0}(x \oplus y, x \oplus w)$. We have: $d_{\geq 0}(x \oplus y, z \oplus w) \leq d_{\geq 0}(x \oplus y, x \oplus w) + d_{\geq 0}(x \oplus w, z \oplus w)$ by corollary 4.2.1 (3) •

COROLLARY: let x, y, z be in X^*. Then

(1) $d^*(x \oplus y, \mathbf{0}) \leq d^*(x, \mathbf{0}) + d^*(y, \mathbf{0})$

(2) $d^*(\rho \odot x, \mathbf{0}) \equiv |\rho| \cdot d^*(x, \mathbf{0})$

(3) $d^*(x,y) \equiv d^*(x \ominus y, \mathbf{0})$

PROOF: we have: $d^*(x \oplus y, \mathbf{0}) \equiv d_{\geq 0}(x^+ \oplus y^+, x^- \oplus y^-)$ and $d^*(x, \mathbf{0}) \equiv d_{\geq 0}(x^+, x^-)$ and $d^*(y, \mathbf{0}) \equiv d_{\geq 0}(y^+, y^-)$ with x^+, x^-, y^+, y^- in $X^*_{\mathbb{R},\geq 0}$. Now apply the lemma. (2) is a trivial consequence of the definitions. For (3) use corollary 4.2.1 (1) to obtain that $d^*(x,y) \equiv d^*(x^+ \oplus y^-, y^+ \oplus x^-) \equiv d^*(x \ominus y, \mathbf{0})$ •

4.2.3* THEOREM: $<(X^*,d^*),\oplus,\odot,\mathbf{0},\|\ \|^*>$ is a normed linear space, such that $d_{\|\ \|^*}=d^*$ coincides with d on X_∞.

PROOF: this is a trivial consequence of definition 4.2.0 and corollary 4.2.2 •

4.2.4* THEOREM: (X,d^*) is halflocated in (X^*,d^*), and strongly so if (X,d) is weakly stable.

PROOF: the proof consists of a number of claims, the first of which is trivial.

$\boxed{\text{claim}}$ (X,d) is best approximable in (X_∞,d).

$\boxed{\text{claim}}$ (X_∞,d) is halflocated in (X_1^*,d^*), and strongly so if (X,d) is weakly stable.

$\boxed{\text{proof}}$ let $x=\bigoplus_{i\leq n}\rho_i\cdot x_i$ be in $X^*_{\mathbb{R},\geq 0}$, where $\sum_i\rho_i\equiv 1$. Let y be in X_∞. Let $\alpha=\inf(\{d(x_j,y)\,|\,j\leq n\}$. Then $d^*(y,x)\equiv\sum_i\rho_i\cdot d(y,x_i)\geq\alpha$. For all $j\leq n$, by triangle inequality: $d^*(y,x)+d^*(y,x_j)\geq d^*(x,x_j)$. From this we obtain: $2\cdot\alpha\geq\inf(\{d^*(x_j,x)\,|\,j\leq n\}$, and so:

$(*)$ $$2\cdot d^*(y,x)\geq\inf(\{d^*(x_j,x)\,|\,j\leq n\})$$

Since y is arbitrary we collect:

$(\star)$ $$\forall m\in\mathbb{Z}\ [\exists j\leq n\ [\,d^*(x_j,x)<3^{m+1}\,]\vee\forall y\in X_\infty\ [\,d^*(y,x)>3^m\,]]$$

This shows that (X_∞,d) is halflocated in (X_1^*,d^*) (with parameter 3). Now suppose (X,d) is weakly stable, then trivially (X_∞,d) is weakly stable. From $(\star)$ we collect:

$(\star\star)$ $$\forall m\in\mathbb{Z}\ \exists(s,z)\in\{0,1\}\times X_\infty\ [(s=0\wedge z\in\{x_i\,|\,i\leq n\}\wedge d^*(x,z)<3^{m+1})\vee (s=1\wedge z=x_0\wedge\forall y\in X\ [\,d^*(y,x)>3^m\,]$$

By $\mathbf{AC}_{01}$ there is a function h from $\mathbb{N}$ to $\{0,1\}\times X_\infty$ realizing $(\star\star)$. Define a Cauchy-sequence $(w_m)_{m\in\mathbb{N}}$ in (X_∞,d) as follows. Let $\beta=\inf(\{d^*(x_j,x)\,|\,j\leq n\})$. Determine $j\leq n$ such that $d^*(x_j,x)<\beta+1$. Put $w_0\underset{D}{=}x_j$, and for $m\in\mathbb{N}$:

$$w_{m+1}\underset{D}{=}\begin{cases} z & \text{if } h(m+1)=(0,z)\\ w_m & \text{if } h(m+1)=(1,x_0)\end{cases}$$

Put $w=d\text{-}\lim(w_m)_{m\in\mathbb{N}}\in\overline{(X_\infty,d)}$. Clearly, by absurdity $\forall i\leq n\ [w\#x_i]$ implies $w\in X_\infty$. Therefore by lemma 3.3.4 w is in X_∞. Using $(\star\star)$ it is easy to see that

$\forall y \in X_\infty \ [d^*(x,w) \leq 9 \cdot d^*(x,y)]$. So then (X_∞, d) is strongly halflocated in (X_1^*, d^*) (with parameter 9) ∘

$\boxed{\text{claim}}$ (X_∞, d) is halflocated in (X^*, d^*), and strongly so if (X,d) is weakly stable.

$\boxed{\text{proof}}$ we use essentially the same argument as in the previous claim. Let x be in X^*. Then $x \equiv x^+ \ominus x^-$ and for all y in X_∞: $d^*(x,y) \equiv d^*(x^+, y \oplus x^-$. Let $\alpha = \sum_i \rho_i^+$ and $\beta = \sum_i \rho_i^-$, then x^+ is in X_α^*. Without loss of generality $\alpha \geq 1+\beta$ (for we can always put $\gamma = \sup(\alpha, 1+\beta)$ and $x' = x \oplus (\gamma - \alpha) \cdot \infty$, then $x \equiv x'$ and x'^+ is in X_γ^* and x'^- is in X_β^* and $\gamma \geq 1+\beta$). Let y in X_∞. Put $z = x^- \oplus (\alpha - 1 - \beta) \cdot \infty$ and put $u = y \oplus z$. Then $d^*(x,y) \equiv d_{eq}(x^+, u)$. Put $u_0 = y$ and $\sigma_0 = 1$, and for $1 \leq k \leq n+1$ put $u_k = x_k$ and $\sigma_k = \rho_k^-$. Finally put $u_{n+2} = \infty$ and $\sigma_{n+2} = \alpha - 1 - \beta$. Then $u = \bigoplus_{k \leq n+2} \sigma_k \cdot u_k$. Let $\tau = (\tau_{i,k})_{i,k \leq n,n+2}$ be a distribution of x^+ in u. Then $\sum_i \tau_{i,0} \equiv \sigma_0 \equiv 1$ therefore $x_{\tau,0}^+ = \bigoplus_i \tau_{i,0} \cdot x_i$ is in X_1^*. Observe that:

$$2 \cdot \textstyle\sum_{i,k} \tau_{i,k} \cdot d(x_i, u_k) = 2 \cdot d^*(x_{\tau,0}^+, y) + 2 \cdot \sum_{i,k;k \geq 1} \tau_{i,k} \cdot d(x_i, u_k)$$

Since $x_{\tau,0}^+$ is in X_1^*, by $(*)$ we find:

$$\begin{aligned} 2 \cdot \textstyle\sum_{i,k} \tau_{i,k} \cdot d(x_i, u_k) &\geq \inf(\{d^*(x_{\tau,0}^+, x_j) \,|\, j \leq n\}) + 2 \cdot \textstyle\sum_{i,k;k \geq 1} \tau_{i,k} \cdot d(x_i, u_k) \\ &\geq \inf(\{d^*(x, x_j) \,|\, j \leq n\}) \end{aligned}$$

The last inequality obtains since τ is also a distribution of x^+ in $x_j \oplus \bigoplus_{i,k;k \geq 1} \sigma_k \cdot u_k$, for $j \leq n$. Also $d^*(x, x_j)$ equals $d_{eq}(x^+, x_j \oplus \bigoplus_{i,k;k \geq 1} \sigma_k \cdot u_k)$, and by definition this latter quantity is less or equal to $d^*(x_{\tau,0}^+, x_j) + \sum_{i,k;k \geq 1} \tau_{i,k} \cdot d(x_i, u_k)$. Since τ is an arbitrary distribution we obtain:

$$2 \cdot d^*(y, x) \geq \inf(\{d^*(x_j, x) \,|\, j \leq n\})$$

Since y is also arbitrary, we reobtain $(\star)$ and $(\star\star)$ for this most general x in X^*. To finish the proof we can now follow the proof of the previous claim ∘

We have that (X,d) is best approximable in (X_∞, d). We have shown that (X_∞, d) is halflocated in (X^*, d^*), and strongly so if (X,d) is weakly stable. The theorem now follows by applying lemma 3.2.5 •

EXAMPLE: we give a Brouwerian counterexample to the statement: 'if (X,d) is a metric space, then (X,d) is located in $(conv(X), d^*)$'. Let $X = \mathbb{N}$ and let d be the metric defined as follows. For $i \neq j$ in $\{0,1,2\}$ put $d(i,j)=1$. For $n \in \mathbb{N}$ put

$$d(n+3,0) = \begin{cases} 0 & \text{if } n+3 \neq k_{99} \\ \frac{1}{2} & \text{if } n+3 = k_{99} \end{cases}$$

Also, if $n+3=k_{99}$ put $d(n+3,1)=\frac{1}{2}=d(n+3,2)$. This completely describes d. Now suppose (X,d) is located in $(conv(X),d^*)$. Then we can compute $\rho = \inf(\{d^*(\bigoplus_{i\in\{0,1,2\}} \frac{1}{3} \odot i, n) \mid n \in \mathbb{N}\}$. If $\rho < \frac{2}{3}$, then $\exists n \in \mathbb{N}\,[n=k_{99}]$ (and $\rho \equiv \frac{1}{2}$). If $\rho > \frac{1}{2}$, then $\forall n \in \mathbb{N}\,[n < k_{99}]$ (and $\rho \equiv \frac{2}{3}$).

REMARK: it is easy to lead the dubious statement above to a contradiction by using **CP**. Notice that in this example there is a d-equivalent metric d' on X such that (X,d') is located in $(X^*,d'^{\star})$. We do not know if this can be proved for an arbitrary metric space (X,d), see also our discussion in section 4.5.

4.2.5* we prove remark 4.0.3 (2), that every weakly stable **AR** is an **AE**.

LEMMA: let (L,d_L) be a linear space. Let $n \in \mathbb{N}$, and let x, y be in $W_n(L,d_L)$. Then $x+y$ is in $W_{2n}(L,d_L)$.

PROOF: we prove the lemma by induction on n.
Basis: $n=0$. Then the lemma is trivially true.
Induction: suppose the lemma holds for $n \in \mathbb{N}$. Let x, y be arbitrary elements of $W_{n+1}(L,d_L)$. Determine w, z in $W_n(L,d_L)$ such that $x \# w$ implies $x \in W_n(L,d_L)$ and $y \# z$ implies $y \in W_n(L,d_L)$. By the induction assumption we have that $w+z$ is in $W_{2n}(L,d_L)$. Clearly $x+z \# w+z$ implies that $x \# w$, which implies that $x \in W_n(L,d_L)$, which in turn implies by the induction assumption that $x+z$ is in $W_{2n}(L,d_L)$. Therefore $x+z$ is in $W_{2n+1}(L,d_L)$. Then $x+y \# x+z$ implies that $y \# z$, which implies that $y \in W_n(L,d_L)$, which in turn implies that $w+y$ is in $W_{2n}(L,d_L)$. But $x+y \# w+y$ then implies that $x \# w$, which implies that $x \in W_n(L,d_L)$, which in turn implies that $x+y$ is in $W_{2n}(L,d_L)$. Therefore we then find that $x+y$ is in $W_{2n+1}(L,d_L)$, on the assumption that $x+y \# x+z$. This shows that $x+y$ is in $W_{2n+2}(L,d_L)$ •

COROLLARY: let (A,d_L) be a convex subspace of a linear space (L,d_L). Then $\overline{(A,d_L)}$ is a convex subspace of $\overline{(L,d_L)}$.

THEOREM: let (X,d) be weakly stable. Then (X,d) is an **AR** iff (X,d) is an **AE**.

PROOF: suppose (X,d) is an **AR**. By theorem 4.2.4 (X,d^*) is strongly halflocated in (X^*,d^*), and so, being an **AR**, there is a retraction π of (X^*,d^*) onto (X,d^*). By theorem 3.3.2 we can extend π to a retraction $\tilde{\pi}$ of $\overline{(X^*,d^*)}$ onto (X,d^*). But $\overline{(X^*,d^*)}$ is a weakly stable convex subset of the normed linear space $\overline{(X^*,d^*)}$, and therefore an **AE** by the Dugundji Extension Theorem 4.1.1. So by remark 4.0.3 (3) (X,d) is an **AE**. By the same remark (1) any **AE** is an **AR** •

4.2.6* REMARK: we can now characterize weakly stable **AE**'s and **AR**'s in a different way. For the previous makes clear that a metric space (X,d) is a weakly stable **AE** iff (X,d) is a weakly stable **AR** iff (X,d) is a retract of a weakly stable convex subspace of a normed linear space. We could have chosen this as a more down-to-earth definition of 'absolute extensor' and 'absolute retract'.

4.3 THE MICHAEL SELECTION THEOREM

4.3.0* in this section we will prove an intuitionistic version of a beautiful theorem by E. Michael [Michael56], which is frequently called the Michael Selection Theorem. Our version of the theorem (thm. 4.3.6) roughly says the following. Let (X,d) be a spreadlike metric space and $(L,\|\ \|)$ a Banach space. Suppose we assign 'in a Lower Semi Continuous manner' to each x in X an inhabited, convex and complete subset $\mathcal{F}(x)$ of $(L,\|\ \|)$. Then there is a continuous function f from (X,d) to $(L,\|\ \|)$ such that for all x in X: $f(x)$ is in $\mathcal{F}(x)$. Such a function f is called a continuous selection for the *set-valued function* $\mathcal{F}$.

In corollary 4.3.6 we strengthen our theorem in the following way. Let (A,d) be strongly sublocated in (X,d). Write $\mathcal{F}\mid_A$ for the restriction of $\mathcal{F}$ to A. Suppose $g:(A,d)\to(L,\|\ \|)$ is a continuous selection for $\mathcal{F}\mid_A$. Then there is a continuous selection f for $\mathcal{F}$ which extends g to (X,d).

Of course we must make the above statements precise. But then we have a powerful tool which will reduce a number of our mathematical problems to ashes, in little to no time at all. We largely follow the development given in [vanMill89, 1.4.6 - 1.4.9], but we must deal with some typically intuitionistic problems.

4.3.1* DEFINITION: let $(X,\mathcal{T})$ and $(Y,\mathcal{T}')$ be two topological spaces. Let $\mathcal{F}$ be a subset of $(X\times Y,\mathcal{T}_{\text{prod}})$. Then $\mathcal{F}$ is called a *set-valued function* from $(X,\mathcal{T})$ to $(Y,\mathcal{T}')$ (notation

$\mathcal{F} : (X,\mathcal{T}) \Rightarrow (Y,\mathcal{T}'))$ iff for all x in X there is a y in Y such that (x,y) is in $\mathcal{F}$. We then write $\mathcal{F}(x)$ for the subset $\{y{\in}Y \mid (x,y){\in}\mathcal{F}\}$, for x in X. A function f from $(X,\mathcal{T})$ to $(Y,\mathcal{T}')$ is called a *selection* for $\mathcal{F}$ iff $f{\subseteq}\mathcal{F}$. Now let $\mathcal{F}$ be a set-valued function from $(X,\mathcal{T})$ to $(Y,\mathcal{T}')$, and let B be a subset of $(Y,\mathcal{T}')$. We put $\mathcal{F}^{\Leftarrow}(B) \underset{D}{=} \{x{\in}X \mid \exists y{\in}Y\ [y{\in}\mathcal{F}(x){\cap}B]\}$. Then $\mathcal{F}$ is called *Lower Semi Continuous* iff for all V in $\mathcal{T}'$: $\mathcal{F}^{\Leftarrow}(V)$ is in $\mathcal{T}$. We usually abbreviate Lower Semi Continuous with LSC.

REMARK: a (continuous) function is a set-valued (LSC) function such that a set in its range consists of mutually equivalent elements.

4.3.2* DEFINITION: let $(X,\mathcal{T})$ be a topological space, and let (L,d_L) be a linear space. Let $\mathcal{F}$ be a set-valued function from $(X,\mathcal{T})$ to (L,d_L). Then $\mathcal{F}$ is *convex* iff for all x in X: $\mathcal{F}(x)$ is a convex subset of (L,d_L). Similarly $\mathcal{F}$ is *complete* iff for all x in X: $\mathcal{F}(x)$ is a complete subset of (L,d_L).

4.3.3* LEMMA: let $\mathcal{F} : (X,d) \Rightarrow (Y,d_Y)$ be LSC. Then

(i) the set-valued function $\mathcal{F}_c : (X,d) \Rightarrow (Y,d_Y)$, defined by $\mathcal{F}_c(x) = \overline{\mathcal{F}(x)}$, is LSC.

(ii) if $f : (X,d) \to (Y,d_Y)$ is continuous, and $r \in \mathbb{R}^+$ such that for all x in X the intersection $B(f(x),r){\cap}\mathcal{F}(x)$ is inhabited, then the function $\mathcal{G} : (X,d) \Rightarrow (Y,d_Y)$ defined by $\mathcal{G}(x) = \overline{B(f(x),r){\cap}\mathcal{F}(x)}$ is LSC.

PROOF: we copy the proof from [vanMill89, lem.1.4.6]. For (i) observe that for all x in X and all open U in (Y,d_Y) we have: $\mathcal{F}(x){\cap}U$ is inhabited iff $\mathcal{F}_c(x){\cap}U$ is inhabited, and so $\mathcal{F}^{\Leftarrow}(U){=}\mathcal{F}_c^{\Leftarrow}(U)$.

Then for (ii) it suffices by (i) to prove that the set-valued function $\mathcal{G}_0 : (X,d) \Rightarrow (Y,d_Y)$ defined by $\mathcal{G}_0(x) \underset{D}{=} B(f(x),r){\cap}\mathcal{F}(x)$ is LSC. To this end let V be open in (Y,d_Y). We show that $\mathcal{G}_0^{\Leftarrow}(V)$ is open in (X,d). Let x be in $\mathcal{G}_0^{\Leftarrow}(V)$. Then there is a y in $(B(f(x),r){\cap}\mathcal{F}(x)) \cap V$. Let $\epsilon{=}r-d(y,f(x))$ and determine a δ in $\mathbb{R}^+$ such that $\delta{<}\epsilon$ and $B(y,\delta){\subseteq}B(f(x),r){\cap}V$. Since $B(y,\frac{1}{2}\delta){\cap}\mathcal{F}(x)$ is inhabited and $\mathcal{F}$ is LSC, we see that $U_0{=}\mathcal{F}^{\Leftarrow}(B(y,\frac{1}{2}\delta))$ is a neighborhood of x in (X,d). Also, $U_1{=}f^{-1}(B(f(x),\frac{1}{2}\delta))$ is a neighborhood of x in (X,d), since f is continuous. Put $U{=}U_0{\cap}U_1$.

$\boxed{\text{claim}}$ $U{\subseteq}\mathcal{G}_0^{\Leftarrow}(V)$.

$\boxed{\text{proof}}$ let z be in U. Since z is in U_0 we can find a w in $B(y, \frac{1}{2}\delta) \cap \mathcal{F}(z)$. Also $f(z)$ is in $B(f(x), \frac{1}{2}\delta)$. Consequently $d(w, f(z)) \leq d(w, f(x)) + d(f(x), f(z))$. But $d(w, f(x)) \leq d(w, y) + d(y, f(x)) < \frac{1}{2}\delta + r - \epsilon$, which gives us that $d(w, f(z)) < r$. Therefore w is in $(B(f(z), r) \cap \mathcal{F}(z)) \cap V$, which is precisely $\mathcal{G}_0^{\Leftarrow}(z) \cap V$. We see that z is in $\mathcal{G}_0^{\Leftarrow}(V)$ ○ •

4.3.4 LEMMA: let $(L, \| \|)$ be a normed linear space and let (X, d) be a spreadlike metric space. Let $\mathcal{F} : (X, d) \Rightarrow (L, \| \|)$ be convex and LSC. Let $r \in \mathbb{R}^+$. Then there is a continuous function f from (X, d) to $(L, \| \|)$ such that: $\forall x \in X \; \exists y \in \mathcal{F}(x) \; [\, d_{\| \|}(f(x), y) < r \,]$

PROOF: let $(b_n)_{n \in \mathbb{N}}$ be dense in $(L, \| \|)$. Then $\mathcal{U} = (\mathcal{F}^{-1}(B(b_n, \frac{1}{2}r)))_{n \in \mathbb{N}}$ is an enumerable open cover of (X, d). By corollary 3.1.3 combined with lemma 3.1.2, there is a partition of unity $(p_m)_{m \in \mathbb{N}}$ on (X, d) such that for all $m \in \mathbb{N}$: ${p_m}^{-1}((0,1]) \subseteq \mathcal{F}^{\Leftarrow}(B(b_m, \frac{1}{2}r))$. Define a continuous function f from (X, d) to $(L, \| \|)$ by:

$$f(x) \underset{D}{=} \sum_{m \in \mathbb{N}} p_m(x) \cdot b_m$$

Now let $x \in X$. There is an $M \in \mathbb{N}$ such that for all $m \in \mathbb{N}$: $m > M$ implies $p_m(x) \equiv 0$. Let $K \in \mathbb{N}$ such that $(\sup(\{d_{\| \|}(b_i, b_j) \mid i, j \leq M\}) + r) < K$. Determine a decidable subset A of $\{0, \ldots, M\}$ such that for all $i \in A$: $p_i(x) > 0$ and $\sum_{i \in A} p_i(x) > 1 - \frac{r}{2K}$. Since for all $i \in A$: $p_i(x) > 0$, we can determine a finite sequence $(y_i)_{i \in A}$ of elements of L such that for all $i \in A$: $y_i \in B(b_i, \frac{1}{2}r) \cap \mathcal{F}(x)$. Let i_0 be the smallest element of A. Put

$$y = \sum_{i \in A} p_i(x) \cdot y_i + (1 - \sum_{i \in A} p_i(x)) \cdot y_{i_0}$$

Then y is a convex combination of elements of $\mathcal{F}(x)$, therefore y is in $\mathcal{F}(x)$. Since $\| \|$ is a norm we find:

$$d_{\| \|}(f(x), y) \leq \sum_{i \in A} p_i(x) \cdot \tfrac{1}{2}r + (1 - \sum_{i \in A} p_i(x)) \cdot K$$

By our choice of A this means that $d_{\| \|}(x, y) < r$ •

4.3.5* LEMMA: let $\mathcal{F} : (X, d) \Rightarrow (Y, d_Y)$ be complete and LSC. Let (A, d) be strongly sublocated in (X, d), and f a continuous selection for $\mathcal{F} \mid_A$, then:

(i) for all $x \in X$ there exist $x_A \in A$ and $x_f \in Y$ satisfying

(1) $x \# x_A$ implies $\exists \rho \in \mathbb{R}^+ \; \forall a \in A [d(x, a) > \rho]$

(2) $x_f \# f(x_A)$ implies $x \# x_A$

(3) $x_f \in \mathcal{F}(x)$

(ii) the function $\mathcal{G} : (X, d) \Rightarrow (Y, d_Y)$ defined by

$$\mathcal{G}(x) = \overline{\{y \in \mathcal{F}(x) \mid x \in A \to y \equiv f(x)\}}$$

is LSC.

PROOF: **a**d (i): (1) is nothing but the definition of strongly sublocated (see 3.2.2). So from now on let $x \in X$ and let $x_A \in A$ satisfy (1).
For (2) we observe that $f(x_A) \in \mathcal{F}(x_A)$ and that $\mathcal{F}$ is LSC. This gives a sequence $(\delta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ such that for all $n \in \mathbb{N}$: $\delta_n < 2^{-n}$, and for all $y \in B(x_A, \delta_n)$: there is a z in $\mathcal{F}(y) \cap B(f(x_A), 2^{-n})$. Let $t \in \mathcal{F}(x)$. We have:

$$(\star) \quad \forall n \in \mathbb{N}\ \exists (m, y) \in \{0,1\} \times Y\ [\,(m=0 \wedge d(x, x_A) < \delta_n \wedge y \in \mathcal{F}(x) \cap B(f(x_A), 2^{-n})) \vee (m=1 \wedge d(x, x_A) > \tfrac{1}{2}\delta_n) \wedge y = t)\,]$$

So by $\mathbf{AC}_{01}$ there is a function $h : \mathbb{N} \to \{0,1\} \times Y$ realizing $(\star)$. Define $t_0 \underset{D}{=} t$, and for $n \in \mathbb{N}$:

$$t_{n+1} \underset{D}{=} \begin{cases} y_{n+1} & \text{if } h(n+1) = (0, y_{n+1}) \\ t_n & \text{if } h(n+1) = (1, t) \end{cases}$$

Trivially, $(t_n)_{n\in\mathbb{N}}$ is d_Y-Cauchy. Let x_f be the d_Y-limit. Suppose $x_f \# f(x_A)$. Then clearly $x \# x_A$. For (3) simply notice that $\mathcal{F}$ is complete and that for all $n \in \mathbb{N}$, t_n is in $\mathcal{F}(x)$.

ad (ii): first we must show that $\mathcal{G} : (X, d) \Rightarrow (Y, d_Y)$. For this it suffices to check that x_f is in $\mathcal{G}(x)$ (notations as above). Now we must prove $\mathcal{G}$ LSC. Suppose U is open in (Y, d_Y) and $y \in \mathcal{G}(x) \cap U$, meaning $x \in \mathcal{G}^{\Leftarrow}(U)$. We have to come up with an η in $\mathbb{R}^+$ such that $B(x, \eta) \subseteq \mathcal{G}^{\Leftarrow}(U)$. Let $\epsilon \in \mathbb{R}^+$ such that $B(y, \epsilon) \subseteq U$. We will consider several (sub)cases, which are not mutually exclusive (but we can always decide: case $\alpha.1$ or case $\alpha.2$).

$\boxed{\text{case 1}}$ $d_Y(y, f(x_A)) > \frac{1}{4}\epsilon$
then since x_A satisfies (i)(2) and $y \in \mathcal{G}(x)$ we must have $x \# x_A$ So by (i)(1) there is a $\rho \in \mathbb{R}^+$ such that for all $z \in B(x, \rho)$: $\forall a \in A\ [\,z \# a\,]$ which implies $\mathcal{G}(z) = \mathcal{F}(z)$. Since $\mathcal{F}$ is LSC there is $\gamma \in \mathbb{R}^+$ such that for all $z \in B(x, \gamma)$ we can find a $w \in \mathcal{F}(z) \cap U$. Clearly now $B(x, \min(\rho, \gamma)) \subseteq \mathcal{G}^{\Leftarrow}(U)$.

$\boxed{\text{case 2}}$ $d_Y(y, f(x_A)) < \frac{1}{2}\epsilon$
determine $\delta \in \mathbb{R}^+$ such that for all $a \in A \cap B(x_A, \delta)$: $f(a) \in B(f(x_A, \frac{1}{2}\epsilon))$. Determine $\rho \in \mathbb{R}^+$ such that for all $z \in B(x_A, \rho)$ we can find a $w \in \mathcal{F}(z) \cap B(f(x_A, \frac{1}{2}\epsilon))$. (Remember f is continuous and $\mathcal{F}$ is LSC).
Put $\gamma = \min(\delta, \rho)$.

$\boxed{\text{case 2.1}}$ $d(x, x_A) > \frac{1}{2}\gamma$
then we are done as in case 1.

$\boxed{\text{case 2.2}}$ $d(x, x_A) < \gamma$
now we hold: $B(x, \gamma - d(x, x_A)) \subseteq \mathcal{G}^{\Leftarrow}(U)$. For suppose $z \in B(x, \gamma - d(x, x_A))$, then $d(z, x_A) < \gamma$. Let $z_A \in A$ and $z_f \in Y$ satisfy (i) (1), (2), and (3) for z.

$\boxed{\text{case 2.2.1}}$ $d(z, z_A) > \frac{1}{2}(\gamma - d(x, x_A) - d(z, x))$
then $\forall a \in A\ [z \# a]$ which implies $\mathcal{G}(z) = \mathcal{F}(z)$. Since $d(z, x_A) < \gamma \leq \rho$ there is $w \in \mathcal{F}(z) \cap B(f(x_A, \frac{1}{2}\epsilon)$ so w is in $\mathcal{G}(z) \cap B(f(x_A, \frac{1}{2}\epsilon) \subset \mathcal{G}(z) \cap U$. So z is in $\mathcal{G}^{\Leftarrow}(U)$.

$\boxed{\text{case 2.2.2}}$ $d(z, z_A) < (\gamma - d(x, x_A) - d(z, x))$
then $d_Y(f(z_A), f(x_A)) < \frac{1}{2}\epsilon$ since $d(z, x) < \gamma \leq \rho$

$\boxed{\text{case 2.2.2.1}}$ $d_Y(z_f, f(z_A)) > \frac{1}{2}\,(\frac{1}{2}\epsilon - d_Y(f(z_A), f(x_A)))$
then $\forall a \in A\ [z \# a]$ and we are done as in case 2.2.1.

$\boxed{\text{case 2.2.2.2}}$ $d_Y(z_f, f(z_A)) < (\frac{1}{2}\epsilon - d_Y(f(z_A), f(x_A)))$
but then $d_Y(z_f, f(x_A)) < \frac{1}{2}\epsilon$ so $d_Y(z_f, y) < \epsilon$ so z_f is in U. Also, by (i)(3), z_f is in $\mathcal{G}(z)$. Therefore z is in $\mathcal{G}^{\Leftarrow}(U)$.

In all of the above cases we have produced an η in $\mathbb{R}^+$ such that $B(x, \eta) \subseteq \mathcal{G}^{\Leftarrow}(U)$. Since these cases cover all possibilities, $\mathcal{G}$ is LSC •

REMARK: this lemma is the constructive version of [vanMill89, lemma 1.4.8.]. The difference between the two reveals the greater attention which must be paid to details, in constructive mathematics. Notice that $\forall x \in X\ [\mathcal{G}(x) \subseteq \mathcal{F}(x)]$.

4.3.6 THEOREM: (Michael Selection Theorem) let $(L, \|\ \|)$ be a Banach space, and (X, d) a spreadlike metric space. Let $\mathcal{F} : (X, d) \Rightarrow (L, \|\ \|)$ be complete, convex, and LSC. Then there is a continuous selection f for $\mathcal{F}$.

PROOF: it suffices to prove the theorem in the case that $X=\sigma$, a spread. Using $\mathbf{DC}_1$ we shall construct a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions from (σ,d) to $(L,\|\ \|)$ such that for all $n\in\mathbb{N}$ and $x\in\sigma$:

(i) $d_{\|\ \|}(f_n(x),f_{n+1}(x))<2^{-n+1}$

(ii) there is an $a_n\in\mathcal{F}(x)$ such that $d_{\|\ \|}(f_n(x),a_n)<2^{-n}$

First apply lemma 4.3.4 with $r=1$ to find a continuous $\tilde{f}:(\sigma,d)\to(L,\|\ \|)$ such that for all $x\in\sigma$ there is an a in $B(f(x),2^{-0})\cap\mathcal{F}(x)$. By lemma 3.0.3 (using $\mathbf{AC}_{10}$), without loss of generality $\tilde{f}$ is a spread-function.

Now let $n\in\mathbb{N}$ and suppose g is in σ_ω such that:

($\star$) g is a continuous spread-function from (σ,d) to $(L,\|\ \|)$ and for all x in σ there is an a in $B(g(x),2^{-n})\cap\mathcal{F}(x)$.

Define $\mathcal{F}_g:(\sigma,d)\Rightarrow(L,\|\ \|)$ by

$$\mathcal{F}_g(x)\underset{D}{=}\overline{\mathcal{F}(x)\cap B(g(x),2^{-n})}$$

Then $\mathcal{F}_g$ is complete, convex, and LSC by lemma 4.3.3. By another appeal to lemma 4.3.4 (with $r=2^{-n-1}$) we find a continuous $\tilde{g}:(\sigma,d)\to(L,\|\ \|)$ such that for all $x\in\sigma$ there is an a in $B(\tilde{g}(x),2^{-n-1})\cap\mathcal{F}_g(x)$. Then we have: $d_{\|\ \|}(g(x),\tilde{g}(x))<2^{-n+1}$ for all $x\in\sigma$. Also, by lemma 3.0.3 (using $\mathbf{AC}_{10}$), without loss of generality $\tilde{g}$ is a spread-function.

Define for $n\in\mathbb{N}$ a subset A_n of σ_ω putting $A_n=\{\gamma\in\sigma_\omega\mid\gamma$ realizes ($\star$) for $n\}$. Put $A=\bigcup_{n\in\mathbb{N}}A_n$, and let R be the subset of $A\times A$ given by:

$$R=\{(\gamma,\delta)\in A\times A\mid\exists n\in\mathbb{N}\,[\,(\gamma,\delta)\in A_n\times A_{n+1}\wedge\forall x\in\sigma\,[d(\gamma(x),\delta(x))<2^{-n+1}]\,]\}$$

Then by our reasoning above we find:

($\star\star$) $\tilde{f}\in A\wedge\forall\alpha\in A\,\exists\beta\in A\,[\,(\alpha,\beta)\in R\,]$

By $\mathbf{DC}_1$ there is a sequence $(f_n)_{n\in\mathbb{N}}$ in A such that $f_0=\tilde{f}$ and for each $n\in\mathbb{N}$: (f_n,f_{n+1}) is in R. Clearly the sequence $(f_n)_{n\in\mathbb{N}}$ satisfies (i) and (ii) above. By (i) and the completeness of $(L,\|\ \|)$, this sequence converges to a continuous $f:(\sigma,d)\to(L,\|\ \|)$. From (ii) and the completeness of $\mathcal{F}$, we collect that $f(x)\in\mathcal{F}(x)$ for all $x\in\sigma$ •

COROLLARY: let (A,d) be strongly sublocated in (X,d), and let f be a continuous selection for $\mathcal{F}\mid_A$. Then there is a continuous selection g for $\mathcal{F}$ which extends f.

PROOF: define $\mathcal{G}$ as in lemma 4.3.5. $\mathcal{G}$ is complete, convex, and LSC by lemma 4.3.5 itself. So we can apply the theorem to find a continuous selection g for $\mathcal{G}$. A fortiori g is a continuous selection for $\mathcal{F}$ which extends f •

4.3.7* we will shed just a little light on the use of the Michael theorem in Bishop's school in the next section.

4.4 STRONG CONTINUITY

4.4.0* the difficulty, especially for Bishop's school, in proving the Michael theorem lies in finding a partition of unity subordinate to a given (enumerable) open cover. Such a partition is needed for proving lemma 4.3.4. This explains the limitation to spreadlike spaces in our version 4.3.6. In the situation where all occurring covers are seen to be per-enumerable (e.g. using lemma 3.1.4), this limitation is no longer necessary, and the Michael theorem becomes applicable in Bishop's school as well. For an example we need a few definitions.

DEFINITION: let (X,d) be a metric space. We define a metric d_p on $X\times\mathbb{R}^+$ as follows: let (x,ϵ) and (y,δ) be in $X\times\mathbb{R}^+$. Then $d_p((x,\epsilon),(y,\delta)) \underset{D}{=} \sup(d(x,y),d_{\mathbb{R}}(\epsilon,\delta))$. We write $B_p((x,\epsilon),\rho)$ for the subset $\{(y,\delta)\mid d_p((x,\epsilon),(y,\delta))<\rho\}$ of $X\times\mathbb{R}^+$. Next let f be a continuous function from (X,d) to (Y,d_Y), another metric space. Let g be a function from $(X\times\mathbb{R}^+,d_p)$ to $(\mathbb{R}^+,d_{\mathbb{R}})$. We call g a *modulus of continuity* for f iff for all x, y in X and all ϵ in $\mathbb{R}^+$: $d(x,y)<g(x,\epsilon)$ implies $d_Y(f(x),f(y))<\epsilon$. We say that f is *strongly continuous* iff there is a continuous modulus of continuity for f.

One fairly easily proves the following. The composition of two strongly continuous functions is strongly continuous. Therefore the sum and the product of two strongly continuous functions are strongly continuous (when this sum and/or product are defined). Also, if we have a uniformly convergent sequence of strongly continuous functions, then the limit function is strongly continuous.

In [vanMill89,1.4.13] the Dugundji theorem 4.1.1 (for normed linear spaces) is derived from the Michael theorem 4.3.6. This approach is also valid intuitionistically if we limit ourselves to spreadlike spaces, as explained above. In Bishop's school the approach can be used for extending a strongly continuous f, since all occurring open covers are then seen to be per-enumerable. This involves quite some work though, and the result is less than the Dugundji theorem presented in 4.1.1. Still it might suggest that especially in Bishop's school strong continuity could be of interest. Intuitionistically, strong continuity is also interesting, but mostly for functions defined on non-spreadlike spaces. For intuitionistically we have that a continuous function from a spreadlike metric space to another metric space is strongly continuous. This is theorem 4.4.2, for the proof of which we use the Michael theorem!

Already in [Veldman82, sect.6] it is proved, using only $\mathbf{AC}_{10}$ (and $\mathbf{AC}_{01}$), that every weak function from $([0,1], d_{\mathbb{R}})$ to $(\mathbb{R}, d_{\mathbb{R}})$ is strongly continuous. We reobtain and extend this result, using in addition only $\mathbf{DC}_1$, by combining theorem 4.4.2 with the second corollary in 3.3.12.

4.4.1* DEFINITION: define a homeomorphism $h : (\mathbb{R}^+, d_{\mathbb{R}}) \hookrightarrow\!\!\!\!\twoheadrightarrow (\mathbb{R}, d_{\mathbb{R}})$ by setting

$$h(\alpha) \underset{D}{=} \begin{cases} 2 - \frac{1}{\alpha} & \text{for } \alpha \leq 1 \\ \alpha & \text{for } \alpha \geq 1 \end{cases}$$

which determines h completely.

REMARK: notice that h and h^{-1} preserve convexity of subsets of $\mathbb{R}$ and $\mathbb{R}^+$ respectively.

4.4.2 THEOREM: every continuous function from a spreadlike metric space to another metric space is strongly continuous.

PROOF: using the Michael theorem 4.3.6. Let f be a continuous function from a spreadlike metric space (X, d) to (Y, d_Y), another metric space. We must show that there is a continuous modulus g for f. Define

$$\mathcal{G}((x,\epsilon)) \underset{D}{=} \{\delta \in \mathbb{R}^+ \mid \exists \gamma \in \mathbb{R}^+ \, \forall (y,\rho) \in B_p((x,\epsilon),\gamma) \; [\forall z \in B(y,\delta) \, [d_Y(f(y), f(z)) < \rho]\,]\}$$

$\boxed{\text{claim}}$ $\mathcal{G} : (X \times \mathbb{R}^+, d_p) \Rightarrow (\mathbb{R}^+, d_{\mathbb{R}})$

$\boxed{\text{proof}}$ clearly if $(x,\epsilon) \equiv (y,\delta)$ then $\mathcal{G}((x,\epsilon)) = \mathcal{G}((y,\delta))$. Let $(x,\epsilon) \in X \times \mathbb{R}^+$, we prove there is a $\delta \in \mathcal{G}((x,\epsilon))$. For let $\eta \in \mathbb{R}^+$, $\eta < \frac{1}{3}\epsilon$ be such that for all $y \in X$: $d(x,y) < \eta$ implies $d_Y(f(x), f(y)) < \frac{1}{3}\epsilon$. Now take $\delta = \frac{1}{2}\eta$. To see that δ is in $\mathcal{G}((x,\epsilon))$, let

$y \in B(x,\delta)$ and $z \in B(y,\delta)$. Then y and z are in $B(x,\eta)$ so $f(y)$ and $f(z)$ are in $B_Y(f(x),\frac{1}{3}\epsilon)$, so $d_Y(f(y),f(z)) < \frac{2}{3}\epsilon$. So for all $(y,\rho) \in X \times \mathbb{R}^+ : d_p((x,\epsilon),(y,\rho)) < \delta$ implies $d(x,y) < \delta$, which implies: for all $z \in B(y,\delta) : d_Y(f(y),f(z)) < \frac{2}{3}\epsilon < \rho$ (since $d_{\mathbb{R}}(\epsilon,\rho) < \delta < \frac{1}{6}\epsilon$). So $\delta \in \mathcal{G}((x,\epsilon))$ ∘

$\boxed{\text{claim}}$ $\mathcal{G}$ is LSC

$\boxed{\text{proof}}$ let U be open in $\mathbb{R}^+$, and suppose $(x,\epsilon) \in \mathcal{G}^{\Leftarrow}(U)$, meaning there is a δ in $\mathcal{G}((x,\epsilon)) \cap U$. This gives us in turn a γ in $\mathbb{R}^+$ such that

$$\forall (y,\rho) \in B_p((x,\epsilon),\gamma)\ [\forall z \in B(y,\delta)\,[d_Y(f(y),f(z)) < \rho]\,]\}$$

Now let $(y,\rho) \in B_p((x,\epsilon),\gamma)$. We wish to prove that δ is in $\mathcal{G}((y,\rho))$. Consider

$$(y',\rho') \in B_p(\,(y,\rho),\gamma - d_p((x,\epsilon),(y,\rho))\,) \ \subseteq\ B_p((x,\epsilon),\gamma)\,.$$

and let $z \in B(y',\delta)$. Then clearly $d_Y(f(y'),f(z)) < \rho'$ since (y',ρ') is in $B_p((x,\epsilon),\gamma)$. (y',ρ') being arbitrary, this means δ is in $\mathcal{G}((y,\rho))$. For there is a γ' in $\mathbb{R}^+$ such that

$$\forall (y',\rho') \in B_p((y,\rho),\gamma')\ [\forall z \in B(y',\delta)\,[d_Y(f(y'),f(z)) < \rho']\,]\}$$

(take $\gamma' = \gamma - d_p((x,\epsilon),(y,\rho))$). Since (y,ρ) is arbitrary, this means: δ is in $\mathcal{G}(\tilde{y},\tilde{\rho}))$ for all $(\tilde{y},\tilde{\rho})$ in $B_p((x,\epsilon),\gamma)$. But then $B_p((x,\epsilon),\gamma) \subseteq \mathcal{G}^{\Leftarrow}(U)$. Since (x,ϵ) is arbitrary, $\mathcal{G}$ is LSC ∘

$\boxed{\text{claim}}$ $\mathcal{G}((x,\epsilon))$ is convex for all (x,ϵ) in $X \times \mathbb{R}^+$

$\boxed{\text{proof}}$ $\delta \in \mathcal{G}((x,\epsilon))$ implies $(0,\delta] \subseteq \mathcal{G}((x,\epsilon))$ ∘

Define $\mathcal{F} : (X \times \mathbb{R}^+, d_p) \to (\mathbb{R}, d_{\mathbb{R}})$ by putting (see 4.4.1)

$$\mathcal{F}((x,\epsilon)) \underset{D}{=} \overline{h(\mathcal{G}((x,\epsilon)))}$$

$\boxed{\text{claim}}$ $\mathcal{F}$ is complete, convex and LSC

$\boxed{\text{proof}}$ combine the previous claims with lemma 4.3.3, and remark 4.4.1 ∘

By the Michael theorem 4.3.6 there is a $\tilde{g} : (X \times \mathbb{R}^+, d_p) \to (\mathbb{R}, d_{\mathbb{R}})$ which is a continuous selection for $\mathcal{F}$. Define $g : (X \times \mathbb{R}^+, d_p) \to (\mathbb{R}^+, d_{\mathbb{R}})$ by

$$g \underset{D}{=} h^{-1} \circ \tilde{g}$$

Clearly g is a continuous selection for $\mathcal{G}$, and cannot as such escape being a continuous modulus for f •

4.5 TOPOLOGICALLY HALFLOCATED SUBSPACES OF (σ, d)

4.5.0 in the first two sections of this chapter we rely on the (strongly) halflocatedness of various subsets of various metric spaces. Then in section 4.3 we prove a theorem (more precise: corollary 4.3.6) which requires only that a certain subspace be strongly sublocated in its (spreadlike) mother-space. This observation leads us to some sort of grand finale. Recall (3.2.2) that we consider (A, d) to be topologically (strongly) halflocated in (X, d) iff there is a d-equivalent metric d' on (X, d) such that (A, d') is (strongly) halflocated in (X, d'). In this section we will prove, amongst others, that a weakly stable (A, d) is strongly sublocated in a spreadlike metric space (X, d) iff (A, d) is topologically strongly halflocated in (X, d). Of course, one of these implications is trivial. To prove the other we will just about need all our previous results.

4.5.1* PROPOSITION: if π is a retraction of (X, d) on (A, d) then there is a d-equivalent metric d_π such that

(i) $d_\pi \mid_{A \times A} \equiv d$

(ii) (A, d_π) is best approximable in (X, d_π), in fact $\forall x \in X\ \forall a \in A\ [\, d_\pi(x, \pi(x)) \leq d_\pi(x, a)\,]$.

(iii) if (B, d) is halflocated in (A, d), then (B, d_π) is halflocated in (X, d_π).

(iv) if in addition (X, d) is complete, then d_π is strongly d-equivalent.

PROOF: define d_π as follows:

$$d_\pi(x, y) \underset{D}{=} \tfrac{1}{2} \cdot (\, d(x, y) + d(\pi(x), \pi(y)) + \mid d(x, \pi(x)) - d(y, \pi(y)) \mid) \ .$$

$\boxed{\text{claim}}$ d_π is a metric.

$\boxed{\text{proof}}$ the only nontrivial concern is the triangle inequality. This however follows easily from the fact that the triangle inequality is satisfied by the three functions $d(x,y)$, $d(\pi(x),\pi(y))$ and $|\,d(x,\pi(x)) - d(y,\pi(y))\,|$ $\circ$

Now (i) is a triviality, (ii) follows from the triangle inequality for d since $d_\pi(x,\pi(x)) = d(x,\pi(x))$ whereas $d_\pi(x,a) = \frac{1}{2}\cdot(d(x,a) + d(a,\pi(x)) + d(x,\pi(x)))$ for $a \in A$. Then we deduce (iii) from (i), (ii) and lemma 3.2.5. Finally, if (X,d) is complete then consider a Cauchy-sequence $(x_n)_{n\in\mathbb{N}}$ in (X,d), say with d-limit $x \in X$. It is an easy consequence of the continuity of π in x that $(x_n)_{n\in\mathbb{N}}$ is also d_π-Cauchy. Any d_π-Cauchy-sequence being d-Cauchy trivially, this finishes (iv) $\bullet$

4.5.2 THEOREM: let (A,d) be strongly sublocated in a spreadlike metric space (X,d). Then (A,d) is topologically halflocated in (X,d), and strongly so if (A,d) is weakly stable.

PROOF: using the previous proposition and theorem 4.3.6. It suffices to prove the theorem for the case that X is a spread σ. We depart from $\overline{(X^*,d^*)}$, in which we consider the subspace $(X \cup \overline{A^*}, d^*)$. By theorem 3.0.2 $\overline{(A^*,d^*)}$ is spreadlike, therefore $(X \cup \overline{A^*}, d^*)$ is spreadlike.

$\boxed{\text{claim}}$ $\overline{(A^*,d^*)}$ is strongly sublocated in $(X \cup \overline{A^*}, d^*)$.

$\boxed{\text{proof}}$ Let x be in X. Determine y in A such that $x \# y$ implies $\exists n\in\mathbb{N}\ \forall a\in A\ [\,d(x,a) > 2^{-n}\,]$. But then, by definition of d^* (see 4.2.0), $x \# y$ implies $\exists n\in\mathbb{N}\ \forall a\in A^*\ [\,d^*(x,a) > 2^{-n}\,]$. For x in $\overline{(A^*,d^*)}$ there is nothing to prove $\circ$

Next, define a set-valued $\mathcal{F} : (X \cup \overline{A^*}, d^*) \Rightarrow \overline{(A^*,d^*)}$ by putting, for x in $(X \cup \overline{A^*}, d^*)$:

$$\mathcal{F}(x) \underset{D}{=} \overline{(A^*,d^*)}$$

Clearly $\mathcal{F}$ is complete, convex and LSC. Moreover, $id_{\overline{(A^*,d^*)}}$ is a continuous selection for the restriction of $\mathcal{F}$ to $\overline{(A^*,d^*)}$. Since $\overline{(A^*,d^*)}$ is a Banach space we can apply theorem 4.3.6 to find a continuous function π from $(X \cup \overline{A^*}, d^*)$ to $\overline{(A^*,d^*)}$ such that that π restricts to the identity on $\overline{(A^*,d^*)}$. That is, π is a retraction of $(X \cup \overline{A^*}, d^*)$ onto $\overline{(A^*,d^*)}$.

Finally, define d_π as in the proof of proposition 4.5.1. We have that (A,d) is halflocated in $\overline{(A^*,d^*)}$ by theorem 4.2.4 and lemma 3.2.2. By (iii) of proposition 4.5.1 we may conclude that (A, d_π) is halflocated in $(X \cup \overline{A^*}, d_\pi)$ so a fortiori in (X, d_π). By the same proposition, d_π is d-equivalent. This shows that (A, d) is topologically halflocated in (X, d).

Now if (A, d) is weakly stable, then (A, d_π) is weakly stable by theorem 3.3.2, and strongly sublocated in (X, d_π) since 'strongly sublocated in' is a topological relation. By (i) we can determine $D \in \mathbb{N}$ such that:

$$\forall x \in X\ \forall n \in \mathbb{N}\ [\exists a \in A\, [d_\pi(x,a) < D^{-n+1}] \vee \forall a \in A\, [d_\pi(x,a) > D^{-n}]]$$

First suppose there is an $n \in \mathbb{N}$ such that $\forall a \in A\ [d_\pi(x,a) > D^{-n}]$. Determine $s \in \mathbb{N}$ and $b \in A$ such that $d_\pi(x,b) < D^{-s+2}$ whereas $\forall a \in A\ [d_\pi(x,a) > D^{-s}]$. Then clearly $\forall a \in A\ [d_\pi(x,b) \leq D^2 \cdot d_\pi(x,a)]$. Now determine y in A such that $x \# y$ implies $\exists n \in \mathbb{N}\ \forall a \in A\ [d_\pi(x,a) > D^{-n}]$. Then by our foregoing reasoning we see:

$$(\star) \quad \forall n \in \mathbb{N}\ \exists (m,b) \in \{0,1\} \times A\ [(m=0 \wedge d_\pi(x,y) < D^{-n} \wedge b=y) \ \vee \\ (m=1 \wedge d_\pi(x,y) > D^{-n-1} \wedge \forall a \in A\, [d_\pi(x,b) \leq D^2 \cdot d_\pi(x,a)]]$$

By $\mathbf{AC}_{01}$ there is a function h from $\mathbb{N}$ to $\{0,1\} \times A$ realizing $(\star)$. Define a Cauchy-sequence in (A, d_π) as follows. Put $z_0 = y$ and for $n \in \mathbb{N}$:

$$z_{n+1} \underset{D}{=} \begin{cases} y & \text{if } h(n+1) = (0,y) \\ z_n & \text{if } h(n) \neq (0,y) \\ b & \text{if } h(n) = (0,y) \text{ and } h(n+1) = (1,b) \end{cases}$$

Put $z = d_\pi\text{-}\lim(z_n)_{n \in \mathbb{N}}$. Then $z \# y$ implies $z \in A$, so z is in A, since (A, d_π) is weakly stable. Moreover: $\forall a \in A\ [d_\pi(x,z) \leq D^2 \cdot d_\pi(x,a)]$, since for any $a \in A$ the assumption $D^2 \cdot d_\pi(x,a) < d_\pi(x,z)$ leads to contradiction. This shows that (A, d_π) is strongly halflocated in (X, d_π). Therefore (A, d) is topologically strongly halflocated in (X, d) •

4.5.3 as a corollary we obtain the following theorem for complete metric spaces:

THEOREM: let (A, d) be (strongly) traceable in a complete metric space (X, d). Then there is a strongly d-equivalent metric d' such that (A, d') is (strongly) halflocated in (X, d').

PROOF: first let (A, d) be traceable in (X, d). By theorem 3.0.2 (X, d) is spreadlike. By lemma 3.3.13 (A, d) is sublocated in (X, d), so by lemma 3.2.4 $\overline{(A, d)}$ is strongly sublocated in (X, d). Clearly $\overline{(A, d)}$ is weakly stable, so with d_π as in the proof of the previous theorem we find that $\overline{(A, d_\pi)}$ is strongly halflocated in (X, d_π). So (A, d_π) is halflocated in (X,d_π) by lemma 3.2.2. Moreover, by proposition 4.5.1 (iv) d_π is strongly d-equivalent, so it suffices to take d' equal to d_π. Notice that if (A,d) is strongly traceable in (X,d), then (A,d) coincides with $\overline{(A,d)}$ •

4.5.4 we prove a theorem which is very similar to the previous one. First we need a lemma.

LEMMA: let (A,d) be strongly traceable in a metric space (X,d). Then $(W_1(A,d),d)$ is strongly traceable in $(W_1(X,d),d)$.

PROOF: let w be in $(W_1(X,d),d)$. Determine x in (X,d) such that $w\#x$ implies $w\in X$. Determine y in A such that $x\#y$ implies $\forall a\in A\ [x\#a]$. We have:

$$(\star)\quad \forall n\in\mathbb{N}\ \exists (s,t,z)\in\{0,1\}\times\{0,1\}\times A\ [\,(s=0 \wedge d(w,x)<2^{-n} \wedge z=y)\vee \\ (s=1 \wedge w\#x \wedge (w\#z \to \forall a\in A\,[w\#a]) \wedge \\ ((t=0\wedge d(w,z)<2^{-n}) \vee (t=1\wedge w\#z))\,]$$

By $\mathbf{AC}_{01}$ there is a function h from $\mathbb{N}$ to $\{0,1\}$ realizing $(\star)$. Determine a function h_0, h_1 from $\mathbb{N}$ to $\{0,1\}$ and a function h_2 from $\mathbb{N}$ to A such that for all $n\in\mathbb{N}$: $h(n)=(h_0(n),h_1(n),h_2(n))$. Define a Cauchy-sequence $(z_n)_{n\in\mathbb{N}}$ in (A,d) by putting $z_0=y$ and for $n\in\mathbb{N}$:

$$z_{n+1} \underset{D}{=} \begin{cases} y & \text{if } h_0(n+1)=0 \\ y & \text{if } h_0(n+1)=1 \text{ and } h_0(n)=0 \text{ and } h_1(n+1)=1 \\ h_2(n+1) & \text{if } h_0(n+1)=1 \text{ and } h_0(n)=0 \text{ and } h_1(n+1)=0 \\ z_n & \text{else} \end{cases}$$

Put $z=d\text{-}\lim(z_n)_{n\in\mathbb{N}}$. Then $z\#y$ implies $z\in A$, therefore z is in $W_1(A,d)$. But $w\#z$ implies $\forall a\in A\ [w\#a]$. Then $w\#z$ implies $\forall a'\in W_1(A,d)\ [w\#a']$. For let $w\#z$ and let a' be in $W_1(A,d)$. Determine b in A such that $a'\#b$ implies $a'\in A$. Then $w\#b$, so $w\#a'$ or $a'\#b$. The last case implies $a'\in A$, and so $w\#a'$ in both cases •

COROLLARY: let (A,d) be strongly traceable in a spreadlike metric space (X,d). Then $\overline{(A,d)}$ is strongly sublocated in $\overline{(X,d)}$.

PROOF: we first show that $\overline{(A,d)}$ is strongly traceable in $\overline{(X,d)}$. Let x be in $\overline{(X,d)}$. Determine $n\in\mathbb{N}$ such that x is in $W_n(X,d)$. By the lemma combined with a trivial induction argument, $W_n(A,d)$ is strongly traceable in $W_n(X,d)$. Determine y in $W_n(A,d)$ such that $x\#y$ implies $\forall a\in W_n(A,d)\ [x\#a]$. We prove by induction that for all $m\in\mathbb{N}$: $x\#y$ implies $\forall a\in W_m(A,d)\ [x\#a]$.

Basis: $m = 0$. Trivially true.
Induction: let $m \in \mathbb{N}$ such that $x \# y$ implies $\forall a \in W_m(A,d)\ [x \# a]$. Suppose $x \# y$. Let z be in $W_{m+1}(A,d)$. Determine w in $W_m(A,d)$ such that $z \# w$ implies $z \in W_m(A,d)$. Since $x \# y$, we have $x \# w$ by the induction assumption. Therefore $x \# z$ or $z \# w$. The last case implies $z \in W_m(A,d)$, and so $x \# z$ in both cases.

So we see that $\overline{(A,d)}$ is strongly traceable in $\overline{(X,d)}$. But $\overline{(X,d)}$ is weakly stable and spreadlike by theorem 3.3.11, so by lemma 3.3.13 $\overline{(A,d)}$ is strongly sublocated in $\overline{(X,d)}$ •

Theorem: let (A,d) be strongly traceable in a spreadlike metric space (X,d). Then $\overline{(A,d)}$ is topologically strongly halflocated in $\overline{(X,d)}$.

proof: by the previous corollary we have that $\overline{(A,d)}$ is strongly sublocated in $\overline{(X,d)}$. Now apply theorem 4.5.2 •

4.5.5 the previous theorem offers a (limited) refinement of the Dugundji theorem 4.1.1, which we formulate thus.

Theorem: let (B, d_L) be a weakly stable convex subspace of a locally convex linear space (L, d_L). Let (A,d) be strongly traceable in a spreadlike metric space (X,d). Let f be a continuous function from (A,d) to (B, d_L). Then there is a continuous extension of f to (X,d).

proof: by corollary 4.5.4 we have that $\overline{(A,d)}$ is strongly sublocated in $\overline{(X,d)}$. By theorem 3.3.5 (i) we can extend f to a continuous function $\tilde{f}$ from $\overline{(A,d)}$ to (B, d_L). By theorem 4.5.4 there is a d-equivalent metric d' such that $\overline{(A,d')}$ is strongly halflocated in $\overline{(X,d')}$. Then $\tilde{f}$ is a continuous function from $\overline{(A,d')}$ to (B, d_L). By the Dugundji theorem 4.1.1 we can extend $\tilde{f}$ to $\overline{(X,d')}$. This extension restricted to (X,d') is the desired extension of f •

remark: notice that the Dugundji theorem does not require any of the spaces involved to be spreadlike. The advantage of the previous theorem (for spreads) is that the condition 'strongly traceable in' is more easily verified than the condition 'topologically (strongly) halflocated in'.

4.5.6* we would of course be happy if 'topologically (strongly) halflocated in' would correspond to 'topologically (strongly) located in', even if we could only prove this for metric spreads.

However, we have been quite unsuccessful both in our attempts to prove such correspondance and in our attempts to find a Brouwerian counterexample.

More generally, notice that the Dugundji theorem does not require any of the spaces involved to be spreadlike. Also, for example we do not know in general how to find for a metric space (X, d), a d^*-equivalent metric d' on X^*, such that (X, d') is located in $(conv(X), d')$, see example 4.2.4. Such being our predicament we feel that, for the time being, 'topologically (strongly) halflocated in' is well worth the trouble.

This finishes our discussion of the different concepts of locatedness of subspaces in their mother-space.

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LIST OF SYMBOLS

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SAMENVATTING

Dit proefschrift handelt over intuïtionistische topologie. De inleiding bestaat uit een summiere geschiedenis, een summiere bespreking van de school van Bishop, en een synopsis van de hoofdstukken èèn tot en met vier. In hoofdstuk nul geven we in kort bestek de benodigde voorkennis weer. In hoofdstuk èèn bouwen we een algemeen-topologisch begrippenapparaat op dat ons van dienst is in de verdere hoofdstukken. In hoofdstuk twee bekijken we verwijderingstopologieën op spreidingen. Hieronder vallen bijvoorbeeld alle kompakte topologische ruimten. In hoofdstuk drie concentreren we ons op metrische ruimten. Hoofdstuk vier is tenslotte gewijd aan funktionaaltopologie, dat wil zeggen topologie waarin kontinue funkties centraal staan. Voor een uitgebreidere samenvatting verwijzen we naar de inleiding.

CURRICULUM VITAE

Born in Amsterdam on May 29, 1965, I am a child of Teddie Besse and Kees Waaldijk, and sibling of Kees and Barbara. From 1972 to 1974 we lived in Msambweni, a small village in Kenya, where we children attended the primary school and received additional education from my mother.

In 1976 I began secondary school in Boxmeer (Elzendaalcollege, VWO). From 1981 to 1982 I was an exchange student at the high school in Minnetonka, Minnesota (USA). Subjects included oil painting, photography, calligraphy and lettering, creative writing and modern political issues. I also studied classical guitar with John Schubert at the MacPhail Institute of the Arts in Minneapolis. In 1983 I finished the VWO in Boxmeer.

I began studying medicine in Nijmegen in the same year (propedeuse 1984). In 1984 I also took up mathematics in Nijmegen; my first course in analysis was taught by Arnoud van Rooij and Wim Veldman. In 1985 I was accepted as an art student in the Akademie voor Beeldende Kunsten in Utrecht (propedeuse 1986). Art being my grand wish, I dropped mathematics and medicine. Hardly two years later I was so disappointed in the ABK Utrecht, that I decided to come back to Nijmegen to finish my math studies. This decision was mainly motivated by the friendly small-scale atmosphere which prevails in the Mathematisch Instituut in Nijmegen.

December 1987 I met Suzan Spijkerman.

I finished my math studies cum laude in 1991; my Master's thesis' supervisor was Wim Veldman. The remaining four months of the year I had a small job as a journalist for the university newspaper K.U.Nieuws. In January 1992 I started as AIO in mathematics (a partly research, partly teaching job for PhD-students), with Wim Veldman as my supervisor.

In 1993 I took up spiritual training[1] with my spiritual master Parthasarathi Rajagopalachari (better known as Chari). Suzan and I were spiritually married by Chari on June 16; we married legally on July 13. On October 30 of the same year Suzan gave

[1]Sahaj Marg (the Natural Path)

birth to our first child Nora. On October 9 of 1995 our second child Femke was born.

Because of two ‘parenthood leaves’ my AIO-contract has been prolonged until June 16 of this year 1996. After this, who knows?