Natural Topology

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Preface to the second edition

In the second edition, we have rectified some omissions and minor errors from the first edition. Notably the composition of natural morphisms has now been properly detailed, as well as the definition of (in)finite-product spaces. The bibliography has been updated (but remains quite incomplete). We changed the names 'path morphism' and 'path space' to 'trail morphism' and 'trail space', because the term 'path space' already has a well-used meaning in general topology.

Also, we have strengthened the part of applied mathematics (the APPLIED perspective). We give more detailed representations of complete metric spaces, and show that natural morphisms are efficient and ubiquitous. We link the theory of star-finite metric developments to efficient computing with morphisms. We hope that this second edition thus provides a unified framework for a smooth transition from theoretical (constructive) topology to applied mathematics.

For better readability we have changed the typography. The Computer Modern fonts have been replaced by the Arev Sans fonts. This was no small operation (since most of the symbol-with-sub/superscript configurations had to be redesigned) but worthwhile, we believe. It would be nice if more fonts become available for LaTeX, the choice at this moment is still very limited.

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Summary

We develop a simple framework called ‘natural topology’, which can serve as a theoretical and applicable basis for dealing with real-world phenomena. Natural topology is tailored to make pointwise and pointfree notions go together naturally. As a constructive theory in BISH, it gives a classical mathematician a faithful idea of important concepts and results in intuitionism.

Natural topology is well-suited for practical and computational purposes. We give several examples relevant for applied mathematics, such as the decision-support system Hawk - Eye, and various real-number representations.

We compare classical mathematics (CLASS), intuitionistic mathematics (INT), recursive mathematics (RUSS), Bishop-style mathematics (BISH) and formal topology, aiming to reduce the mutual differences to their essence. To do so, our mathematical foundation must be precise and simple. There are links with physics, regarding the topological character of our physical universe.

Any natural space is isomorphic to a quotient space of Baire space, which therefore is universal. We develop an elegant and concise ‘genetic induction’ scheme, and prove its equivalence on natural spaces to a formal-topological induction style. The inductive Heine-Borel property holds for ‘compact’ or ‘fanlike’ natural subspaces, including the real interval $[\alpha, \beta]$. Inductive morphisms respect this Heine-Borel property, inversely. This partly solves the continuous-function problem for BISH, yet pointwise problems persist in the absence of Brouwer’s Thesis.

By inductivizing the definitions, a direct correspondence with INT is obtained which allows for a translation of many intuitionistic results into BISH. We thus prove a constructive star-finitary metrization theorem which parallels the classical metrization theorem for strongly paracompact spaces. We also obtain non-metrizable Silva spaces, in infinite-dimensional topology. Natural topology gives a solid basis, we think, for further constructive study of topological lattice theory, algebraic topology and infinite-dimensional topology.

The final section reconsiders the question of which mathematics to choose for physics. Compactness issues also play a role here, since the question ‘can Nature produce a non-recursive sequence?’ finds a negative answer in $\mathbf{CT}_{\text{phys}}$. $\mathbf{CT}_{\text{phys}}$, if true, would seem at first glance to point to RUSS as the mathematics of choice for physics. To discuss this issue, we wax more philosophical. We also present a simple model of INT in RUSS, in the two-player game LIfE.

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CHAPTER ZERO

Introduction

In this chapter we introduce the subject of natural topology, starting from the natural sciences. The basis for observations and measurements in science seems inescapably to be one of ever-increasing refinement. We do not obtain finished real numbers, but only finite approximations of ever-increasing exactitude.

This type of 'constructive' considerations has become increasingly important with the advent of the computer. The translation of theoretical classical mathematics to applied mathematics is often problematic. Constructive mathematics incorporates finite approximations in its theory, thus providing a smooth theoretical framework for applied mathematics.

Also important: natural topology gives a mathematical model of the real-world topology that we encounter when measuring. This we believe to be relevant for physics.

Last but not least, we believe natural topology to be relevant for the foundations of (constructive) mathematics.

0.0 INTRODUCTION

0.0.0 **Background and motivation of this paper** For a historical background and motivation of this paper, we refer the reader to the appendix, section A.1. Section A.2 of the appendix holds some nice mathematical examples which should help clarify our approach. For readability, most proofs are given in the appendix in section A.3. Constructive axioms and concepts are given and discussed in section A.4, additional remarks can be found in section A.5, and the bibliography is in section A.6.

Summarizing: don't skip the appendix!

0.0.1 **Introduction to natural topology** Imagine an engineer taking measurements of some natural physical phenomenon. With ever-increasing precision steps she arrives at ever more precise approximations of certain real numbers. In this process she may come across two measurements which at the outset could still indicate the same real number, yet, when more precision is attained, are seen to be really apart.

The interesting thing about this description lies in the hidden meaning of the word 'real number'. Usually this meaning is taken for granted, with an intuitive image of the real line as 'foundation'. But in mathematics —the precision language of science— the real numbers are commonly defined as equivalence classes of Cauchy-sequences of rational numbers. Then, later, a metric topology can be defined where the basic open sets are the open rational intervals. This topology is optional, as a system the real numbers are usually viewed to exist 'on their own'.

Compared to the situation that the engineer finds herself in, the above mathematical approach is exactly the other way round. For the engineer first and only encounters a finite number of shrinking rational intervals (the measurements), and then regards these finite measurements as an approximation to a real number (getting better all the time if one can apply more and more precision).

From a topological view, the engineer comes across the topology of rational intervals before ever seeing a real number. It turns out that many problems of translating theoretical mathematics into practical applications hinge around this reversal of approach to real numbers and their natural topology. So let us go into this matter a bit more.

0.0.2 **Classical mathematics** In theoretical mathematics, theorems about the real numbers are often proved in a way which 'disregards' the topology. For instance, consider a theorem asserting $\forall x \in \mathbb{R} \exists y \in \mathbb{R}[D(x, y)]$. Consider a reasonably algorithmic proof even which has the following form: for $x < 0$ do A and for $x \geq 0$ do B. Looking at it from a topological standpoint, one could say such a proof is 'discontinuous' in 0. When an engineer looks to implement the theorem say on a computer fed by real-time data, the problem arises immediately that for real-time data x the distinction $x < 0$ or $x \geq 0$ cannot always be made. On data hovering around 0, the 'method' supplied by the theoretical proof might lead to a non-terminating program.

0.0.3 **Constructive mathematics** These types of problems have partly motivated the development of *constructive mathematics*, especially of course since the advent of the computer. Constructive mathematics is a branch of mathematics in which theorems are proved in such a way that the translation of the proof to a working program should be immediate (however in general no claim to efficiency is made).

The surprise for mathematicians and engineers alike is that many theorems from 'classical' mathematics can be shown to be 'non-constructive', meaning that they can never be translated into a working program. In essence, then, these theorems are simply untrue in constructive mathematics. This questioning of hitherto 'solid' theory has led to quite some reluctance amongst theoretical mathematicians to adopt a 'constructive' outlook.

The downside of this reluctance has been that applicability issues are left to ad-hoc solutioneering, whereas a clearer and more efficient mathematical approach is possible. We hope to give part of such an approach in this paper. Amongst other results, this approach yields a simple topological foundation to some practical issues arising from different representation methods of the real numbers (e.g. Cauchy-sequences, decimal notation floating points, binary notation floating points, interval arithmetic and others).

0.0.4 **Our framework: pointwise as well as pointfree** The topological framework which we present should also be of theoretical interest, both classically and constructively. For instance, in our (classically valid) framework we present a pathwise connected metric space, which is not arcwise connected. (How this can be? Read and see.) For constructive mathematics, the development of topology has been tackled in different ways. Our approach seems a simple and elegant alternative to the avenues explored so far.

For large parts of mathematics the concept of ‘points’ seems natural and elegant. Therefore we focus on this concept. Still, we will show that points arise naturally from ’pointfree’ topological constructions. We believe that our ‘pointfree’ machinery is simple, compared to the framework of formal topology and pointfree topology. This simplicity might look restrictive in the sense that our machinery leads us only a little further than ‘separable topological T_1-spaces’. On the other hand, that is a vast class of spaces. Bishop (the founder of BISH) even went as far as saying that non-separable spaces are a form of pseudogenerality which is to be avoided in constructive math. We hope that by keeping things simple, we can explain the relevance of constructive topology to the ‘working class’ classical/applied mathematician.

From the foundational perspective, another advantage of keeping things simple is that axiomatic and conceptual assumptions become clear. These assumptions also reflect on physics. As an example we like to state already here that topological compactness of the unit real interval $[0, 1]$ turns out to be an independent axiom. There is a perfectly acceptable and beautiful model of the real numbers in which $[0, 1]$ is not topologically compact (only very trivial spaces are topologically compact in this model). To our knowledge no one has put forth a convincing argument why reality is not better modeled by this non-compact real model than by the ‘standard’ real model. But the present monograph gives a handhold for the discussion, we believe. In fact we hope that the monograph can serve the foundations of (constructive) mathematics in general.

Topological spaces always exist in conjunction with continuous mappings between them. We define different types of such mappings, to deal with lattice and tree structures which arise naturally from topological investigations. We think that our choices are suited for both theoretical and practical (computational) purposes.

In order to achieve a constructively valid framework, our logic and our proof methods are constructive. This will not impact greatly on our presentation, which seems quite natural if one keeps in mind that all results should be implementable on a computer. For a precise axiomatic account the reader can read the appendix A.4.

0.0.5 **Whom it may concern** In the light of the above, we think this paper is of interest from four different perspectives: applied mathematics & computer science (APPLIED), general mathematics (GENERAL), constructive foundations of mathematics (CONSTRUCTIVE) and foundations of physics (PHYSICS).

CHAPTER ONE

Natural Topology

In this chapter we give the definition of 'natural space', starting with the topology and obtaining the points in the process. The natural real numbers are a prime example. Natural morphisms between natural spaces are defined, and shown to be continuous. Conversely, continuous functions going to 'basic neighborhood spaces' can be represented by morphisms. Still, for CLASS the equivalence structure determined by isomorphisms is finer than the equivalence structure determined by homeomorphisms. Natural topology is seen to resemble intuitionistic topology.

Natural Baire space and Cantor space are defined. We show that the class of natural spaces is large, containing (representations of) every complete separable metric space.

From the APPLIED perspective we discuss the natural topology of binary, ternary and decimal reals. We also look at the well-known Cantor function, and examine the line-calling decision-support system Hawk-Eye which is used in professional tennis.

1.0 BASIC DEFINITIONS AND THE NATURAL REALS

1.0.0 Topology first, points later From the previous introduction our mathematical challenge becomes clear. Namely how to define the real numbers - and more generally a (separable) topological space – *starting* with the topology, and obtaining the points of the topological space in the process.

1.0.1 Dots and points We turn to an intuitive picture of the situation that our previously defined engineer finds herself in. This picture briefly runs as follows. Our engineer in fact encounters only ‘dots’ or ‘specks’, which for the sake of our mathematical argument we think of as being arbitrarily refinable. Two ‘dots’ defined by different processes might at some approximation be seen to definitely lie apart, in which case they represent different (real) numbers. But if the dots are still overlapping at some approximative state, then our engineer cannot tell whether the dots represent different numbers or not.

So the information of dots lying apart gives more tractability than the information of dots overlapping. The first situation allows a definite conclusion at a finite state (two different real numbers) whereas the second situation still hovers around two possible conclusions. Therefore we will define our spaces using the apartness properties of dots, more than overlap properties.

In the above intuitive picture, the dots play the central role, and the real numbers arise only as an idealization. Namely, a real number arises as the intersection of an infinite ‘ever-shrinking’ sequence of dots. ‘Ever-shrinking’ can be defined in terms of the apartness of dots, but this turns out to be less convenient than introducing a second notion regarding dots, namely: ‘being a refinement of’, which behaves like a partial order $\preceq$ on the countable collection of dots. In accordance with our intuitive picture, when $a \preceq b$, then a is a ‘refinement’ of b and represents a ‘smaller’ dot, contained in b.[1]

1.0.2 Pre-natural spaces From the previous introduction we distill the basic mathematical setting: we have a countable set V of *basic dots* of a *natural*

[1] Still, we should keep in mind that basic dots arise in the course of a process of measuring points. We will see that when creating pointwise mappings -or in other words, when looking at transformations- it can be meaningful to distinguish between the unit interval $[0,1]$ as a refinement of $[0,2]$ and the ‘same’ unit interval $[0,1]$ as a refinement of $[-1,1]$.

topological space $(\mathcal{V}, \mathcal{T}_{\#})$ which we build with a number of definitions in this section. Along with the definitions we give some explanations and examples.

DEFINITION: A pre-natural space is a triple $(V, \#, \preceq)$ where V is a countable[2] set of *basic dots* and $\#$ and $\preceq$ are binary relations on V, satisfying the properties following below. Here $\#$ is a *pre-apartness* relation (expressing that two dots lie apart) and $\preceq$ is a *refinement* relation (expressing that one dot is a refinement of the other, and therefore contained in the other).

(i) The relations $\#$ and $\preceq$ are *decidable* on the basic dots.[3]

(ii) For all $a, b \in V$: $a \# b$ (‘a is apart from b’) if and only if $b \# a$. Pre-apartness is symmetric.

(iii) For all $a \in V$: $\neg(a \# a)$. Pre-apartness is antireflexive.

(iv) For all $a, b, c \in V$: if $a \preceq b$ (‘a refines b’) then $c \# b$ implies $c \# a$. Pre-apartness is $\succeq$-monotone.

(v) The relation $\preceq$ is a partial order, so for all $a, b, c \in V$: $a \preceq a$ and if $a \preceq b \preceq c$ then $a \preceq c$, and if $a \preceq b \preceq a$ then $a = b$. Refinement is reflexive, transitive and antisymmetric.

For basic dots we write $a \approx b$ (‘a touches b’) iff $\neg(a \# b)$. Then $\approx$ is the decidable complement of $\#$. (END OF DEFINITION)

REMARK: For the motivating example of the real numbers, the basic dots can be thought of as the rational intervals.[4] Two rational intervals $[a, b]$ and $[c, d]$ are said to be *apart*, notation $[a, b] \# [c, d]$, iff either $d < a$ or $b < c$. $[a, b]$ *refines* $[c, d]$, notation $[a, b] \preceq [c, d]$, iff $c \leq a$ and $b \leq d$. (Also see 1.0.8 and appendix A.5.0) (END OF REMARK)

1.0.3 Points arise from shrinking sequences Of course one idea is to turn to infinite shrinking sequences (of dots), in order to arrive at points. Looking at our example of rational intervals we see that we need to impose a ‘sufficient

[2] A set S is *countable* iff there is a bijection from $\mathbb{N}$ to S, and *enumerable* iff there is a surjection from $\mathbb{N}$ to S.

[3] This means we have a finite procedure to decide the relation. Consider e.g. two rational intervals $[a, b]$ and $[c, d]$, we can decide whether these intervals lie apart or not. We can also decide whether one is a refinement of the other, which in this case is the same as being contained in the other, or not.

[4] Using open or closed intervals both yields the same structure, but working with closed intervals fits the intuitive picture better.

shrinking' condition, otherwise the infinite intersection may contain a whole interval rather than just a point. For an infinite shrinking sequence $\alpha = r_0, r_1, \ldots$ of closed rational intervals ($r_{m+1} \preceq r_m$ for all indices m) to represent a real number, α must 'choose' between each pair of apart rational intervals $[a,b] \# [c,d]$. By which we mean: for each such pair $[a,b] \# [c,d]$, there is an index m such that $r_m \# [a,b]$ or $r_m \# [c,d]$. (We leave it to the reader to verify that this is indeed equivalent to saying that the infinite intersection of $(r_m)_{m \in \mathbb{N}}$ contains just one real number.).

The elegance of this approach is that for an infinite shrinking sequence of dots, the property of 'being a point' can be expressed by an enumerable condition of pre-apartness. There is no need to talk of 'convergence rate' or 'Cauchy-sequence', which both presuppose some metric concept. To define points, we can simply study the real numbers and transfer certain of their nice properties to our general setting.

DEFINITION: A *point* on the pre-natural space $(V, \#, \preceq)$ is an infinite sequence $p = p_0, p_1, p_2 \ldots$ of elements of V that satisfies:

(i) for all indices n we have: $p_{n+1} \preceq p_n$ and there is an index m with $p_m \prec p_n$.

(ii) If $a, b \in V$ and $a \# b$ then there is an index m such that $p_m \# a$ or $p_m \# b$.

Note that any infinite subsequence of p is itself a point (equivalent to p in the natural sense to be defined). The set of all points on $(V, \#, \preceq)$ is denoted by $\mathcal{V}$. (END OF DEFINITION)

Since points are infinite sequences, the set $\mathcal{V}$ is generally not enumerable (but all points in $\mathcal{V}$ could be equivalent).

1.0.4 **Apartness on points** The points of our pre-natural space $(V, \#, \preceq)$ are defined, but clearly we obtain many points which are in some sense equivalent (see our example of rational intervals). The constructive approach to an equivalence relation is to look at its strong opposite, namely an *apartness* (see below for the standard properties of an apartness).
Therefore it is convenient to extend $\#$ to points in $\mathcal{V}$, and also define when points 'belong' to dots, in the obvious way:

DEFINITION: For $p = p_0, p_1, \ldots, q = q_0, q_1, \ldots \in \mathcal{V}$ and $a \in V$:

(i) $a \# p$ and $p \# a$ iff $a \# p_m$ for some index m.

(ii) $p \# q$ iff $p_n \# q_n$ for some index n.

(iii) $p \equiv q$ iff $\neg(p \# q)$.

(iv) $p \prec \alpha$ iff $p_m \prec \alpha$ for some index m. This relation is also referred to as 'α is a beginning of p' or 'p begins with α' or 'p belongs to α'.

(v) We write $[\alpha]$ for the set of all points p such that α is a beginning of p. Notice that $[\alpha]$ is not necessarily closed under $\equiv$. We write $[\![\alpha]\!]$ for the $\equiv$-closure of $[\alpha]$.

(END OF DEFINITION)

In terms of complexity, $\#$ is a Σ^1_0-property, whereas $\equiv$ is a Π^1_0-property. This reflects that apartness of two sequences of dots can be seen at some finite stage, but equivalence of two such sequences is an infinite property. Therefore apartness is better suited for constructive and computational purposes (also see appendix A.5.1).

To see that $\#$ is indeed an apartness (and that therefore $\equiv$ is an equivalence relation), notice that for all $p = p_0, p_1 \ldots$ and $q = q_0, q_1 \ldots$ and $r = r_0, r_1 \ldots$ in $\mathcal{V}$:

(1) $\neg(p \# p)$ (*anti-reflexivity*).

(2) $p \# q$ implies $q \# p$ (*symmetry*)

(3) if $p \# q$ then $p \# r$ or $q \# r$ (*co-transitivity*). (If $p \# q$ then there is n with $p_n \# q_n$, therefore by definition of points (1.0.3(ii)) we can find an index m with: $r_m \# p_m$ or $r_m \# q_m$, so $p \# r$ or $q \# r$).

1.0.5 **Apartness topology is the natural topology** There is a natural topology on the set of points $\mathcal{V}$ of a pre-natural space $(V, \#, \preceq)$. This topology is expressed in terms of apartness and refinement, we call it the *natural topology* and also the *apartness topology* [5], denoted as $\mathcal{T}_\#$. $\mathcal{T}_\#$ is the collection of $\#$-open subsets of $\mathcal{V}$ where $\#$-open is defined thus:

DEFINITION: A set $U \subseteq \mathcal{V}$ is *$\#$-open* iff for each $x \in U$ and each $y \in \mathcal{V}$ we can determine at least one of the following two conditions (they need not be mutually exclusive):

(1) $y \# x$

(2) there is an index m such that $[y_m] = \{z \in \mathcal{V} \mid z \prec y_m\}$ is contained in U.

[5] See appendix A.1.3 for some historical remarks.

When the context is clear we simply say ‘open’ instead of ‘ $\#$ -open’. (END OF DEFINITION)

It follows from this definition that an open set is saturated for the equivalence on points (meaning if U is open, $x \in U$ and $x \equiv y$ then $y \in U$). We leave this to the reader for easy verification. (This also means that we could replace $[y_m]$ with $[\![y_m]\!]$ in (2) above, but in practice this leads to slightly more elaborate proofs). Let us first show that the above indeed defines a topology on $\mathcal{V}$:

$\mathbf{Top_1}$ Clearly the empty set $\emptyset$ and the entire set $\mathcal{V}$ are open.

$\mathbf{Top_2}$ Let $U, W \subseteq \mathcal{V}$ be open sets, we wish to show that $U \cap W$ is open. For this suppose $x \in U \cap W$ and $y \in \mathcal{V}$, we must show: $y \# x$ or there is an index m such that all points beginning with y_m are contained in $U \cap W$. However, since U is open, we can choose case $U(1)$ $y \# x$ or case $U(2)$ there is an index s such that all points beginning with y_s are contained in U. Since W is open, we can also choose case $W(1)$ $y \# x$ or case $W(2)$ there is an index t such that all points beginning with y_t are contained in W. Combining these two choices, we find: $y \# x$ or for $m = \max(s, t)$ all points beginning with y_m are contained in $U \cap W$.

$\mathbf{Top_3}$ Suppose that $U \subseteq \mathcal{V}$ is a set and each $x \in U$ has an open neighbourhood W_x such that $x \in W_x \subseteq U$. We must show that U is open (this is the constructive formulation of ‘an arbitrary union of open sets is open’). For this suppose $x \in U$ and $y \in \mathcal{V}$, we must show: $y \# x$ or there is an index m such that all points beginning with y_m are contained in U. Determine an open neighbourhood W_x such that $x \in W_x \subseteq U$. Since W_x is open, we find: $y \# x$ or there is an index m such that all points beginning with y_m are contained in W_x and therefore in U. We see that U is open.

1.0.6 Natural spaces All the ingredients for our main definition have been prepared. Notice that we did not yet stipulate that each dot should at least contain a point. Also it turns out to be necessary to have a maximal dot, which contains the entire space. These then become the final requirements:

DEFINITION: Let $(V, \#, \preceq)$ be a pre-natural space, with corresponding set of points $\mathcal{V}$ and apartness topology $\mathcal{T}_\#$. An element d of V is called a *maximal dot* iff $a \preceq d$ for all $a \in V$. Notice that V has at most one maximal dot[6], which

[6]Actually, it also makes sense to reverse the $\preceq$-notation, and to consider our maximal dot as being the minimal element, which carries the least information. Then each refinement is ‘larger’ because it carries more information than its predecessor.

if existent is denoted $\bigcirc_V$ or simply $\bigcirc$. $(\mathcal{V}, \mathcal{T}_\#)$ is a *natural space* iff V has a maximal dot and every $a \in V$ contains a point. (END OF DEFINITION)

LEMMA: Let $(\mathcal{V}, \mathcal{T}_\#)$ be a natural space, with corresponding pre-natural space $(V, \#, \preceq)$. Let $a \in V$ be a basic dot. Then the set $\#([a]) = \{z \in \mathcal{V} \mid z \# a\}$ is open in the natural topology.

COROLLARY: For x in $(\mathcal{V}, \mathcal{T}_\#)$, the set $\{w \in \mathcal{V} \mid w \# x\}$ is open in the natural topology. So a set containing one point (up to equivalence) is closed, showing that every natural space is T_1.

PROOF: let x be in $\#([a])$, and let y be in $\mathcal{V}$. We need to show one of the following two conditions:

(1) $y \# x$

(2) there is an index m such that $[y_m] = \{z \in \mathcal{V} \mid z \prec y_m\}$ is contained in $\#([a])$.

Since $x \# a$, by definition there is an index s with $x_s \# a$. Therefore by definition of points there is an index m with $y_m \# x_s$ or $y_m \# a$. In the first case we find that $y \# x$, in the second we find that $[y_m]$ is contained in $\#([a])$. For the corollary, notice that $\{w \in \mathcal{V} \mid w \# x\} = \bigcup_{n \in \mathbb{N}} \{w \in \mathcal{V} \mid w \# x_n\}$. (END OF PROOF)

REMARK: One might think that the lemma shows that in our approach the dots correspond to 'closed' subsets in the topology, but this is not always the case. We can also construct the real numbers as a natural space where the basic dots correspond to open intervals, see the alternative definition in paragraph 1.0.8. In this monograph, different representations of the 'same' natural space are studied also to arrive at computational efficiency (for example, consider floating-point versus interval arithmetic). This is relevant for the APPLIED perspective. (END OF REMARK)

1.0.7 **Other ways to define natural spaces** There are other ways (than the definitions above) to introduce points and spaces. One such way is to simply look at sequences of basic dots which are not necessarily successive refinements, but which all touch (so none are apart) and which fulfill the same condition of 'choosing between each pair of apart dots'. However, the resulting point-space can be easily transformed in an 'equivalent' space in which the points are again given by a refinement condition.

To see this, note that we can always move to new basic dots which are made up of a finite sequence $a_0, \ldots, a_n$ of 'old' basic dots where a_i touches a_j for all $i, j \leq n$. Then a new dot $b = b_0, \ldots, b_m$ refines a new dot $a = a_0, \ldots, a_n$ whenever $m \geq n$ and $b_i = a_i$ for all $i \leq n$.

The above transition from basic dots to finite sequences of basic dots plays a part in our discussion of continuous mappings later on. Two different points may start out differently yet pass through the same basic dot at some later point in time. As we said earlier, we think that for a nice theory of such mappings one should be able to distinguish between $[0,1]$ as a refinement of $[0,2]$ and $[0,1]$ as a refinement of $[-1,1]$. This can be easily realized by looking at the different finite sequences $a = ([0,2],[0,1])$ and $b = ([-1,1],[0,1])$.

1.0.8 The natural real numbers After using the rational intervals as a running example for V, we can now formally define the *natural real numbers* $\mathbb{R}_{\mathrm{nat}}$ as follows:

DEFINITION: Let $\mathrm{R}_{\mathbb{Q}} \underset{D}{=} \{[p,q] | p,q \in \mathbb{Q} | p < q\} \cup \{(-\infty,\infty)\}$. For two rational intervals $[a,b]$ and $[c,d]$ put $[a,b] \#_{\mathbb{R}} [c,d]$ iff ($d < a$ or $b < c$) and put $[a,b] \preceq_{\mathbb{R}} [c,d]$ iff ($c \leq a$ and $b \leq d$). The maximal dot $\bigcirc_{\mathbb{R}}$ is obviously $(-\infty,\infty)$. The points on the pre-natural space $(\mathrm{R}_{\mathbb{Q}}, \#_{\mathbb{R}}, \preceq_{\mathbb{R}})$ are called the *natural real numbers* (also 'natural reals'), the set of natural reals is denoted by $\mathbb{R}_{\mathrm{nat}}$. The corresponding natural topology is denoted by $\mathcal{T}_{\#_{\mathbb{R}}}$. (Also see the remark later in this paragraph).

Next, let $[0,1]_{\mathrm{R}} \underset{D}{=} \{[p,q] | p,q \in \mathbb{Q} | 0 \leq p < q \leq 1\}$, then $([0,1]_{\mathrm{R}}, \#_{\mathbb{R}}, \preceq_{\mathbb{R}})$ is a pre-natural space with corresponding natural space $([0,1]_{\mathrm{nat}}, \mathcal{T}_{\#_{\mathbb{R}}})$ and maximal dot $\bigcirc_{[0,1]} = [0,1]$. (END OF DEFINITION)

THEOREM: $(\mathbb{R}_{\mathrm{nat}}, \mathcal{T}_{\#_{\mathbb{R}}})$ is a natural space which is homeomorphic to the topological space of the real numbers $\mathbb{R}$ equipped with the usual metric topology.

PROOF:
We prove this in the appendix (A.3.0), it is not difficult. Notice that by 'homeomorphism' we mean the usual definition (a continuous function from one space to the other which has a continuous inverse; 'continuous' meaning that the inverse image of an open set is itself open). Also notice that we are a bit free here, since for a classical theorist we should first move to the quotient space of equivalence classes. (END OF PROOF)

REMARK: An interesting alternative definition of $\mathbb{R}$ as a natural space is obtained by changing just very little in the definition. For two rational intervals $[a,b]$ and $[c,d]$ in $\mathsf{R}_{\mathbb{Q}}$ put $[a,b]\,\#^{\circ}_{\mathbb{R}}[c,d]$ iff ($d \leq a$ or $b \leq c$) and put $[a,b] \preceq^{\circ}_{\mathbb{R}} [c,d]$ iff ($c < a$ and $b < d$). Then$(\mathsf{R}_{\mathbb{Q}},\ \#^{\circ}_{\mathbb{R}},\ \preceq^{\circ}_{\mathbb{R}})$ is a pre-natural space, and the corresponding natural space is again homeomorphic to $\mathbb{R}$. But one sees that the basic dots $[a,b]$ now correspond to the *open* real intervals (a,b).[7] $(\mathsf{R}_{\mathbb{Q}},\ \#^{\circ}_{\mathbb{R}},\ \preceq^{\circ}_{\mathbb{R}})$ corresponds to the definition of the formal reals in formal topology (we believe). However, we think compactness is less wieldy in $(\mathsf{R}_{\mathbb{Q}},\ \#^{\circ}_{\mathbb{R}},\ \preceq^{\circ}_{\mathbb{R}})$, which is one reason to stick with $(\mathsf{R}_{\mathbb{Q}},\ \#_{\mathbb{R}},\ \preceq_{\mathbb{R}})$.[8] (END OF REMARK)

From a classical point of view the theorem above however also begs the question: 'If what we get are the same old real numbers, then what did we gain?'. To answer this question we turn to a final important element of natural spaces: morphisms between them.

1.1 NATURAL MORPHISMS

1.1.0 **Why study morphisms?** Insight into natural spaces is gleaned from mappings from one space to another which are structure-preserving to some extent. We define different types of such mappings, calling all of them unimaginatively *natural morphisms*. Each natural morphism defines a continuous function with respect to the natural topology. More surprising, from a classical point of view, is that the structure of natural morphisms gives a finer distinction between natural spaces, than the structure of continuous functions between their corresponding topological spaces. This means that the class of natural morphisms forms an interesting subclass of the class of continuous functions. In 1.2.2 we present a nice condition on a natural space $(\mathcal{V}, \mathcal{T}_{\#})$, which when satisfied guarantees in CLASS, INT and RUSS that a continuous function from another natural space to $(\mathcal{V}, \mathcal{T}_{\#})$ can be represented by a natural morphism, see 1.2.2.

We will show that there is no *isomorphism* between the natural real numbers and the 'natural decimal real numbers', whereas classically these spaces are

[7] It would therefore be better to denote the basic dots as open rational intervals (a,b) under this definition.

[8] The two spaces are isomorphic in a sense yet to be defined.

topologically identical.[9] Another interesting result: the space of 'natural decimal real numbers' turns out to be 'pathwise$_{\mathrm{nat}}$ connected' but not 'arcwise$_{\mathrm{nat}}$ connected'. This is fact the translation of an intuitionistic result. Similarly, many intuitionistic results can be translated to the setting of natural spaces and natural morphisms, providing an alternative classical way to view important parts of intuitionism.

1.1.1 Different representations of the 'same' space In topology, homeomorphisms play a central role. When two spaces are homeomorphic, one can see them as two different representations of the 'same' topological space. Yet there is often an intrinsic interest in these different representations. Consider for example $\mathbb{R}$ and $\mathbb{R}^+$. These are two homeomorphic spaces ($(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$ are even isomorphic topological groups), but we often have use for one or the other representation, depending on context.

In order to build an elegant theory and prove its correctness, we will need to look at many different representations of 'same' natural spaces. As can be expected, in natural topology 'sameness' is induced by a special class of natural morphisms called 'isomorphisms'. Every isomorphism induces a homeomorphism, but the converse is not true in CLASS (see the above example of the natural decimal real numbers).

1.1.2 Natural morphisms 1: refinement morphisms We will distinguish two types of natural morphisms: *refinement morphisms* (denoted $\preceq$-morphisms) and *trail morphisms* (denoted $\wr$-morphisms). The definition of refinement morphisms should pose no surprises in the light of our previous narrative.

When going from one natural space to another, a refinement morphism sends basic dots to basic dots, respecting the apartness and refinement relations, in such a way that 'points go to points'. This means that any $\preceq$-morphism is an order morphism with respect to the partial order $\preceq$.[10]

DEFINITION: Let $(\mathcal{V}, \mathcal{T}_{\#_1})$ and $(\mathcal{W}, \mathcal{T}_{\#_2})$ be two natural spaces, with corresponding pre-natural spaces $(V, \#_1, \preceq_1)$ and $(W, \#_2, \preceq_2)$. Let f be a function from V to W. Then f is called a *refinement morphism* (notation: $\preceq$-morphism) from $(\mathcal{V}, \mathcal{T}_{\#_1})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$ iff for all $a, b \in V$ and all $p = p_0, p_1, \ldots \in \mathcal{V}$:

(i) $f(a) \#_2 f(b)$ implies $a \#_1 b$.

[9] This is interesting for representation issues in computer science.

[10] Not all order morphisms are refinement morphisms though. Our notation '$\preceq$-morphism' can be slightly misleading in this respect.

(ii) $a \preceq_1 b$ implies $f(a) \preceq_2 f(b)$

(iii) $f(p) \underset{D}{=} f(p_0), f(p_1), \ldots$ is in $\mathcal{W}$.

As indicated in (iii) above we will write f also for the induced function from $\mathcal{V}$ to $\mathcal{W}$. The reader may check that (ii) follows from (iii). By (i), a $\preceq$-morphism f from $(\mathcal{V}, \mathcal{T}_{\#_1})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$ respects the apartness/equivalence relations on points, since $f(p) \#_2 f(q)$ implies $p \#_1 q$ for $p, q \in \mathcal{V}$. (END OF DEFINITION)

THEOREM: Let f be a $\preceq$-morphism from $(\mathcal{V}, \mathcal{T}_{\#_1})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$. Then f is continuous.

PROOF: Let U be open in $(\mathcal{W}, \mathcal{T}_{\#_2})$. We must show that $T = f^{-1}(U)$ is open in $(\mathcal{V}, \mathcal{T}_{\#_1})$. For this let $x \in T$, and $y \in \mathcal{V}$. We must show: $x \#_1 y$ or there is an index m such that $[y_m] \subseteq T$. Since $f(x)$ is in U, we can choose case $U(1)$ $f(x) \#_2 f(y)$, then $x \#_1 y$ by (i) above; or case $U(2)$ there is an index m such that $[f(y)_m] \subseteq U$, then by (iii) and (ii) above: $[y_m] \subseteq T$. (END OF PROOF)

1.1.3 Refinement morphisms are computationally efficient

The concept of $\preceq$-morphisms is simple. They have the added advantage of computational efficiency. With a suited 'lean' representation $\sigma_{\mathbb{R}}$ of the natural real numbers, $\preceq$-morphisms from $\sigma_{\mathbb{R}}$ to $\sigma_{\mathbb{R}}$ resemble interval arithmetic, and match the recommendations in [BauKav2009] for efficient exact real arithmetic.

More generally, a continuous function between two 'lean' natural spaces can usually be represented by a $\preceq$-morphism (see thm. 1.2.2 and prp. 2.3.2). Therefore we are interested, already from the APPLIED perspective, in constructing 'lean' representations of natural spaces (called 'spraids'). Spraids turn out to be fundamental for the GENERAL, CONSTRUCTIVE and PHYSICS perspectives as well. To understand the complexities and to prove our framework correct, we need to define trail morphisms and trail spaces.

1.1.4 Natural morphisms 2: trail morphisms

For the most general setting of natural spaces and pointwise topology, $\preceq$-morphisms turn out to be too restrictive. This explains our use for the more involved concept of 'trail morphism' (denoted $\wr$-morphism). Trail morphisms play a necessary role in establishing nice properties of natural spaces. But once these properties have been established, we will primarily use $\preceq$-morphisms (see the previous paragraph). Where $\preceq$-morphisms are defined naturally on basic dots, one can see $\wr$-morphisms as mappings which are naturally defined on points.

Actually, a trail morphism from a natural space $(\mathcal{V}, \mathcal{T}_\#)$ to another space $(\mathcal{W}, \mathcal{T}_{\#_2})$ is given by a refinement morphism from the 'trail space' associated with $(\mathcal{V}, \mathcal{T}_\#)$, to $(\mathcal{W}, \mathcal{T}_{\#_2})$. To define this trail space, we form new basic dots from finite sequences of 'old' basic dots (as described in 1.0.7).

DEFINITION: Let $(\mathcal{V}, \mathcal{T}_\#)$ be a natural space derived from $(V, \#, \preceq)$. Let $n \in \mathbb{N}$, and let $a = a_0 \succeq \ldots \succeq a_{n-1}$ be a shrinking sequence of basic dots in V. The $\prec$*-trail of* a, notation $\overline{\overline{a}}$, is the longest subsequence $a_0 \succ \ldots \succ a_s$ of a.
Let $p = p_0, p_1, \ldots$ be a point in $\mathcal{V}$, then we write $\overline{p}(n)$ for the finite sequence $p_0, \ldots, p_{n-1}$ of basic dots in V. Notice that $p_0 \succeq \ldots \succeq p_{n-1}$, by definition of points. Write $\overline{\overline{p}}(n)$ for the $\prec$-trail of $\overline{p}(n)$. A finite sequence $a = a_0 \succ \ldots \succ a_{n-1}$ of basic dots in V is called a $\prec$*-trail* from a_0 to a_{n-1} of length n, or simply a trail from a_0 to a_{n-1} in $(V, \prec)$. The empty sequence is the unique trail of length 0, and denoted $\bigcirc^*$. The countable set of trails in $(V, \prec)$ is denoted $V^\wr$, notice that $V^\wr = \{\overline{\overline{p}}(n) \mid n \in \mathbb{N}, p \in \mathcal{V}\}$.

Let $a = a_0, \ldots, a_{n-1}$ and $b = b_0, \ldots, b_{m-1}$ be trails in $(V, \prec)$ such that $a_{n-1} \succ b_0$, then we write $a \star b$ for the concatenation $a_0, \ldots, a_{n-1}, b_0 \ldots b_{m-1}$ which is again a trail and so in $V^\wr$. (Hereby $a \star \bigcirc^*$ is defined and equals a.).

The basic dots of our trail space are the trails in $(V, \prec)$. For trails $a = a_0, \ldots, a_{n-1}$ and $b = b_0, \ldots, b_{m-1}$ we put: $a \preceq^* b$ iff there is a trail $c \in V^\wr$ in such that $a = b \star c$. We also put $a \#^* b$ iff $a_{n-1} \# b_{m-1}$. The natural space $(\mathcal{V}^\wr, \mathcal{T}^\wr_\#)$ defined by the pre-natural space $(V^\wr, \#^*, \preceq^*)$ is called the *trail space* of $(\mathcal{V}, \mathcal{T}_\#)$.

Finally, a $\preceq$-morphism f from $(\mathcal{V}^\wr, \mathcal{T}^\wr_\#)$ to another natural space $(\mathcal{W}, \mathcal{T}_{\#_2})$ is called a *trail morphism* (notation $\wr$-morphism) from $(\mathcal{V}, \mathcal{T}_\#)$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$. For a point $p \in \mathcal{V}$ we write $f(p)$ for the point of $\mathcal{W}$ given by $f(\overline{\overline{p}}(0)), f(\overline{\overline{p}}(1)), \ldots$.
(END OF DEFINITION)

REMARK: From the pointwise perspective, one readily sees that $(\mathcal{V}^\wr, \mathcal{T}^\wr_\#)$ is 'just another representation' of $(\mathcal{V}, \mathcal{T}_\#)$. Differences in representation should be filtered out by the concept of 'isomorphism'. This is the main reason for introducing trail morphisms, since $(\mathcal{V}^\wr, \mathcal{T}^\wr_\#)$ is not always $\preceq$-isomorphic to $(\mathcal{V}, \mathcal{T}_\#)$ (for an example consider the natural real numbers). In fact refinement morphisms preserve the lattice-order properties of the basic neighborhood system which is chosen for a specific representation. Due to the presence of an apartness/equivalence relation, these order properties are not always relevant since we can freely add or distract equivalent basic dots to our system with different lattice properties, without essentially changing the point space. Also see A.5.3 for extra comments. (END OF REMARK)

THEOREM: Let $(\mathcal{V}, \mathcal{T}_{\#})$ and $(\mathcal{V}^{\wr}, \mathcal{T}^{\wr}_{\#})$, $(V^{\wr}, \#^*, \preceq^*)$ be as in the above definition. Then

(i) $(V^{\wr}, \#^*, \preceq^*)$ is a pre-natural space and $(\mathcal{V}^{\wr}, \mathcal{T}^{\wr}_{\#})$ is a natural space.

(ii) $(\mathcal{V}^{\wr}, \mathcal{T}^{\wr}_{\#})$ is homeomorphic to $(\mathcal{V}, \mathcal{T}_{\#})$ as a topological space. A homeomorphism is induced by the $\wr$-morphism $\mathbf{id}_{\wr}$ from $(\mathcal{V}, \mathcal{T}_{\#})$ to $(\mathcal{V}^{\wr}, \mathcal{T}^{\wr}_{\#})$ given by $\mathbf{id}_{\wr}(p) = \overline{\overline{p}}(0), \overline{\overline{p}}(1), \overline{\overline{p}}(2), \ldots \in \mathcal{V}^{\wr}$ for $p \in \mathcal{V}$ (as a refinement morphism $\mathbf{id}_{\wr}$ is the identity on $V^{\wr}$, with $\mathbf{id}_{\wr}(a) = a$ for $a \in V^{\wr}$). Its inverse homeomorphism is induced by the $\preceq$-morphism $\mathbf{id}_*$ from $(\mathcal{V}^{\wr}, \mathcal{T}^{\wr}_{\#})$ to $(\mathcal{V}, \mathcal{T}_{\#})$ which is defined by putting $\mathbf{id}_*(\bigcirc^*) = \bigcirc$ and $\mathbf{id}_*(a) = a_n$ for a trail $a = a_0, \ldots, a_n$ in $V^{\wr}$.

(iii) Let f be a $\wr$-morphism from $(\mathcal{V}, \mathcal{T}_{\#})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$. Then f is continuous.

PROOF: Ad (i): this is a straightforward checking of the definitions, which we leave to the reader.

Ad (ii): First we show that $\mathbf{id}_{\wr}$ is continuous.
Let U be open in $(\mathcal{V}^{\wr}, \mathcal{T}^{\wr}_{\#})$. We must show that $T = \mathbf{id}_{\wr}^{-1}(U)$ is open in $(\mathcal{V}, \mathcal{T}_{\#})$. For this let $x \in T$, and $y \in \mathcal{V}$. We must show: $x \# y$ or there is an index m such that $[\![y_m]\!] \subseteq T$. Since $\mathbf{id}_{\wr}(x)$ is in U, we can choose case $U(1)$ $\mathbf{id}_{\wr}(x) \#^* \mathbf{id}_{\wr}(y)$, then there is $n \in \mathbb{N}$ such that $\overline{\overline{x}}(n) \#^* \overline{\overline{y}}(n)$ which implies $x_{n-1} \# y_{n-1}$ and so $x \# y$; or case $U(2)$ there is an index m such that $[\![\overline{\overline{y}}(m+1)]\!] \subseteq U$. Then one easily sees that $[\![y_m]\!] \subseteq T$. For let $z \in [\![y_m]\!]$, then let $z_{y_m} = y_0, \ldots, y_m, z_M, z_{M+1}, \ldots$ be the canonical z-equivalent point such that $\overline{z_{y_m}}(m+1) = \overline{y}(m+1)$, where M is the first index for which $z_M \prec y_m$. Trivially $\mathbf{id}_{\wr}(z_{y_m})$ is in U. Since U is open, U is closed under equivalence, and therefore $\mathbf{id}_{\wr}(z)$ is in U, which shows that z is in T.

The $\preceq$-morphism $\mathbf{id}_*$ is continuous by theorem 1.1.0. That both morphisms are injective[11], and therefore homeomorphisms is trivial.

Ad (iii): Strictly speaking, f is defined as a $\preceq$-morphism $f^{\wr}$ from $(\mathcal{V}^{\wr}, \mathcal{T}^{\wr}_{\#})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$. So as a function from $(\mathcal{V}, \mathcal{T}_{\#})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$, simply notice that $f = f^{\wr} \circ \mathbf{id}_{\wr}$, where $f^{\wr}$ is continuous by theorem 1.1.0 and $\mathbf{id}_{\wr}$ is continuous by (ii) above, and so f is continuous as well. (END OF PROOF)

If f is a $\preceq$-morphism from $(\mathcal{V}, \mathcal{T}_{\#_1})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$, then $f \circ \mathbf{id}_*$ is by definition a $\wr$-morphism from $(\mathcal{V}, \mathcal{T}_{\#_1})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$, which is clearly equivalent to f on $\mathcal{V}$. Therefore we will consider each $\preceq$-morphism to be a $\wr$-morphism as well.

[11] Meaning: $x \# y$ implies $f(x) \# f(y)$.

1.1.5 **Natural morphisms' convention** The difference between the notions $\preceq$-morphism and $\wr$-morphism is often not relevant, which justifies the following:

CONVENTION: If $(\mathcal{V}, \mathcal{T}_{\#_1})$ and $(\mathcal{W}, \mathcal{T}_{\#_2})$ are two natural spaces, and f is a $\preceq$-morphism or a $\wr$-morphism from $(\mathcal{V}, \mathcal{T}_{\#_1})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$, where the difference is irrelevant, then we simply say: f is a *natural morphism* from $(\mathcal{V}, \mathcal{T}_{\#_1})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$, or even more simply: a morphism from $(\mathcal{V}, \mathcal{T}_{\#_1})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$. Only when the difference is relevant will we specify 'refinement morphism' and/ or 'trail morphism'. This happens mostly in technical proofs or in the context of computation, since refinement morphisms are generally more efficient. (END OF CONVENTION)

1.1.6 **Composition of natural morphisms** Given two $\preceq$-morphisms f, g from natural spaces $\mathcal{V}$ to $\mathcal{W}$ and $\mathcal{W}$ to $\mathcal{Z}$ respectively, to form their composition is unproblematic. We leave it to the reader to verify that putting $h(a)=g(f(a))$ for all $a \in V$ defines a $\preceq$-morphism h from $\mathcal{V}$ to $\mathcal{Z}$.

But if f, g are $\wr$-morphisms from natural spaces $\mathcal{V}$ to $\mathcal{W}$ and $\mathcal{W}$ to $\mathcal{Z}$ respectively, then how do we form the composition? Here $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{Z}$ are derived from the pre-natural spaces V, W, Z, with (pre-natural) trail spaces $V^{\wr}, W^{\wr}, Z^{\wr}$ respectively.

One should notice that f is defined as a $\preceq$-morphism from $V^{\wr}$ to W, but can be uniquely lifted to a $\preceq$-morphism $f^{\wr}$ from $V^{\wr}$ to $W^{\wr}$. This is straightforward, for a basic dot $a = a_0, a_1, \ldots, a_{n-1}$ in $V^{\wr}$ we look at $b = f(a_0), f(a_0, a_1), \ldots, f(a_0, a_1, \ldots, a_{n-1})$ and put $f^{\wr}(a)=\overline{\overline{b}}$ (the $\prec$-trail of b) which is a basic dot in $W^{\wr}$ since f is a $\preceq$-morphism.

Therefore the composition of f and g is defined to be the composition $g \circ f^{\wr}$, which is a $\wr$-morphism from $\mathcal{V}$ to $\mathcal{Z}$. Since any $\preceq$-morphism can be thought of as a $\wr$-morphism (trivially), the composition of a $\preceq$-morphism with a $\wr$-morphism can be similarly dealt with. The composition of a $\wr$-morphism with a $\preceq$-morphism directly yields a new $\wr$-morphism.

1.1.7 **Isomorphisms** We can now define a natural parallel to the topological idea of 'homeomorphism'. We will call this parallel 'isomorphism'. Isomorphisms between natural spaces will automatically be homeomorphisms, but classically we can find homeomorphic natural spaces which are non-isomorphic. This shows that our theory enriches CLASS as well.

DEFINITION: Let $(\mathcal{V}, \mathcal{T}_{\#_1})$ and $(\mathcal{W}, \mathcal{T}_{\#_2})$ be two natural spaces, let f be a natural morphism from $(\mathcal{V}, \mathcal{T}_{\#_1})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$. Then f is called an *isomorphism* iff there is a morphism g from $(\mathcal{W}, \mathcal{T}_{\#_2})$ to $(\mathcal{V}, \mathcal{T}_{\#_1})$ such that $g(f(x)) \equiv_1 x$ for all x in $\mathcal{V}$ and $f(g(y)) \equiv_2 y$ for all y in $\mathcal{W}$. An isomorphism f from $(\mathcal{V}, \mathcal{T}_{\#_1})$ to $(\mathcal{V}, \mathcal{T}_{\#_1})$ is called an *automorphism* of $(\mathcal{V}, \mathcal{T}_{\#_1})$, and an *identical automorphism* iff $f(x) \equiv_1 x$ for every $x \in \mathcal{V}$. (END OF DEFINITION)

To see whether certain properties of natural spaces are truly 'natural', we check if they are preserved under isomorphisms.

1.2 FUNDAMENTAL NATURAL SPACES

1.2.0 **Baire space and Cantor space** Baire space ($\mathbb{N}^{\mathbb{N}}$) is fundamental because it is a universal natural space (meaning that every natural space can be thought of as a quotient space of Baire space). From chapter two on, we exploit this to simplify the theory considerably. Cantor space ($\{0,1\}^{\mathbb{N}}$) is likewise a universal 'fan' by which we mean a space generated by a partial order $\preceq$ which is a finitely branching tree. Cantor space can be seen as a universal compact space.

1.2.1 **The class of natural spaces is large** Many spaces can be represented by a natural space. In other words, the class of natural spaces is large. A non-exhaustive and also repetitive list of spaces which can be represented as a natural space:

- every complete separable metric space
- the (in)finite product of natural spaces
- $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, the complex p-adic numbers $\mathbb{C}_p$, $\mathbb{R}^{\mathbb{N}}$, Baire space, Cantor space, Hilbert space $\mathbb{H}$, every Banach space, the space of locally uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$, many other continuous-function spaces, and Silva spaces (see chapter four).

Sometimes, classically defined non-separable spaces (for instance function spaces equipped with the sup-norm) can be constructed under a different metric to become separable. Although the topology is then not equivalent, one can still work with the space constructively as well. For this, one sometimes needs to construct a completion first, to refind the original space as a subset of the completion. Thinking things through, we do not really see a constructive way to define 'workable' spaces other than by going through some enumerably converging process. In this sense we concur with Brouwer. Brouwer's definition of spreads in essence parallels the definition of natural spaces. But unlike Brouwer, we are also engaged in achieving computational efficiency (APPLIED perspective), as well as establishing links between CLASS, INT, RUSS and BISH (and formal topology).

An example of a continuous function space which cannot be represented as a natural space is the space of continuous functions from Baire space to itself. We prove this result (copied from [Vel1981]) using natural morphisms, in 2.4.2. Still, we will see that there is a subset **Mor** of Baire space $\mathbb{N}^{\mathbb{N}}$ such that every $\alpha \in$**Mor** codes a natural morphism from Baire space to itself, and every natural morphism from Baire space to itself is coded by some $\alpha \in$**Mor**.

1.2.2 Basic-open spaces and basic neighborhood spaces Basic dots do not always represent an open set, or even a neighborhood in the apartness topology.[12] Still, so-called 'basic-neighborhood spaces' are fundamental, especially in the context of metric spaces. In CLASS, INT and RUSS every continuous function from a natural space to a basic neighborhood space $(\mathcal{V}, \mathcal{T}_\#)$ can be represented by a natural morphism. The idea is to look at basic dots a which are neighborhoods, meaning $[\![a]\!]$ contains an inhabited open U.

DEFINITION: Let $(\mathcal{V}, \mathcal{T}_\#)$ be a natural space, with corresponding pre-natural space $(V, \#, \preceq)$. Let a be a basic dot, and let $x \in [a]$. Then a is called a *basic (open) neighborhood* of x iff $[\![a]\!]$ is a neighborhood of x (resp. $[\![a]\!]$ is itself open). Now $(\mathcal{V}, \mathcal{T}_\#)$ is called a *basic-open space* iff $[\![a]\!]$ is open for every $a \in V$. $(\mathcal{V}, \mathcal{T}_\#)$ is called a *basic neighborhood space* iff $(\mathcal{V}, \mathcal{T}_\#)$ is isomorphic to a basic-open space. (END OF DEFINITION)

REMARK: 'Basic-open space' is not a 'natural' property, meaning that it is not necessarily preserved under isomorphisms (see 1.0.8, where $(\mathrm{R}_{\mathbb{Q}}, \#^{\circ}_{\mathbb{R}}, \preceq^{\circ}_{\mathbb{R}})$ and $(\mathrm{R}_{\mathbb{Q}}, \#_{\mathbb{R}}, \preceq_{\mathbb{R}})$ are isomorphic, yet only $(\mathrm{R}_{\mathbb{Q}}, \#^{\circ}_{\mathbb{R}}, \preceq^{\circ}_{\mathbb{R}})$ is basic-open). So we 'naturalize' the concept 'basic-open space' to 'basic neighborhood space', which then trivially is preserved under isomorphisms. This 'naturalizing' of desirable properties might seem a bit cheap, but is simply practical. Perhaps basic neighborhood spaces are also definable by the following property, but we leave this as a challenge to the reader. (END OF REMARK)

[12] In contrast to formal topology, where one only works with opens. We believe this to be unwieldy, both computationally and w.r.t. compactness issues.

PROPOSITION: If $(\mathcal{V}, \mathcal{T}_\#)$ is a basic neighborhood space then there is an identical automorphism f of $(\mathcal{V}, \mathcal{T}_\#)$ such that $f(x)_n$ is a basic neighborhood of $f(x)$ for every $x \in \mathcal{V}, n \in \mathbb{N}$ ('there is an identical automorphism which constructs every point as a shrinking sequence of basic neighborhoods of that point').

PROOF: Let (w.l.o.g., see A.5.2) g, h respectively be $\preceq$-isomorphisms between $(\mathcal{V}, \mathcal{T}_\#)$ and the basic-open space $(\mathcal{W}, \mathcal{T}_{\#_2})$. Put $f = h \circ g$, then f is an identical automorphism. Let $x \in \mathcal{V}, n \in \mathbb{N}$, then $\llbracket g(x_n) \rrbracket$ is open in $(\mathcal{W}, \mathcal{T}_{\#_2})$ since $(\mathcal{W}, \mathcal{T}_{\#_2})$ is basic-open. Since h is also a homeomorphism of the topologies, we see that $h(g(x_n)) = f(x)_n$ is a basic neighborhood of $f(x)$. (END OF PROOF)

If $(\mathcal{V}, \mathcal{T}_\#)$ is a basic neighborhood space derived from $(V, \#, \preceq)$, then V contains a neighborhood basis for the natural topology. The converse does not hold in CLASS: see A.2.5 for a space where in CLASS every point has an equivalent which arises as a shrinking sequence of basic neighborhoods of that point, and yet there is no identical automorphism sending each point to such an equivalent shrinking sequence. This illustrates that for natural spaces the information '$\forall x \exists y\ [P(x, y)]$' only becomes effective if we know that there is a morphism f such that '$\forall x\ [P(x, f(x))]$'.[13]

The prime example of a basic neighborhood space is a basic-open space where the basic dots represent open sets (then the identity on V is an identical automorphism which constructs every point as a shrinking sequence of basic neighborhoods of that point). We put forward the main theorem, that in CLASS, INT and RUSS continuous functions from a natural space to a basic neighborhood space can be represented by a natural morphism. Later we show that every complete metric space has a basic-open representation (and therefore in CLASS, INT and RUSS by the corollary below a unique representation (up to isomorphism) as a basic neighborhood space, see 1.2.3).

THEOREM: (in CLASS, INT and RUSS)
Let f be a continuous function from a natural space $(\mathcal{V}, \mathcal{T}_{\#_1})$ to a basic neighborhood space $(\mathcal{W}, \mathcal{T}_{\#_2})$. Then there is a natural morphism g from $(\mathcal{V}, \mathcal{T}_{\#_1})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$ such that for all x in $\mathcal{V}$: $f(x) \equiv_2 g(x)$.

PROOF: The not so easy proof is given in the appendix (A.3.1). In INT the existence of g follows already from the information '$\forall x \exists y [y = f(x)]$' and the fact that every natural space is 'spreadlike' (see 2.2.0). (END OF PROOF)

[13] In INT, the statement '$\forall x \exists y\ [P(x, y)]$' is generally equivalent to 'there is a morphism f such that $\forall x\ [P(x, f(x))]$'. This is precisely the content of the INT-axiom of continuous choice $\mathbf{AC}_{11}$ (see A.4.4, and $^{\mathrm{x}}$27.1 in [Kle&Ves1965]).

COROLLARY: (in CLASS, INT and RUSS) If $(\mathcal{V}, \mathcal{T}_{\#_1})$ and $(\mathcal{W}, \mathcal{T}_{\#_2})$ are two homeomorphic basic neighborhood spaces, then they are isomorphic.

REMARK: The theorem suggests that from a BISH point of view, the concept of ‘natural morphism’ adequately captures the notion of continuous function (under the usual topological definition). To (partly) capture the metric property ‘uniformly continuous on compact subspaces’ we will define ‘inductive morphisms’ later on. (END OF REMARK)

1.2.3 **Complete separable metric spaces are natural** To see that the class of natural spaces is large enough to merit interest, we point out with a theorem below that every complete separable metric space is homeomorphic to a natural space. Therefore every separable metric space is homeomorphic to a subspace of a natural space. Some key examples of spaces which can be constructed as a natural space are $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, the complex p-adic numbers $\mathbb{C}_p$, $\mathbb{R}^{\mathbb{N}}$, Baire space, Cantor space, Hilbert space $\mathbb{H}$, and every Banach space. We prove slightly more, because of our interest in different representations of complete metric spaces:

THEOREM: Every complete separable metric space (X, d) is homeomorphic to a basic-open space $(\mathcal{V}, \mathcal{T}_{\#})$.

PROOF: The rough idea is simple: for a separable metric space (X, d) with dense subset $(a_n)_{n\in\mathbb{N}}$, let for each $n, s\in\mathbb{N}$ a basic dot be the open sphere $B(a_n, 2^{-s}) = \{x\in X \mid d(x, a_n) < 2^{-s}\}$. Then we have an enumerable set of dots V by taking $V = \{B(a_n, 2^{-s}) \mid n, s\in\mathbb{N}\}$. The only trouble now is to define $\#$ and $\preceq$ constructively, since in general for n, m and s, t the containment relation $B(a_n, 2^{-s})\subseteq B(a_m, 2^{-t})$ is not decidable. We leave this technical trouble, which can be resolved using $\mathbf{AC}_{01}$ (countable choice), to the appendix A.3.2. (END OF PROOF)

COROLLARY: In CLASS, INT and RUSS the following holds:

(i) A continuous function f from a natural space $(\mathcal{W}, \mathcal{T}_{\#})$ to a complete metric space (X, d) can be represented by a morphism from $(\mathcal{W}, \mathcal{T}_{\#})$ to a basic neighborhood space $(\mathcal{V}, \mathcal{T}_{\#})$ homeomorphic to (X, d), by theorem 1.2.2.

(ii) A representation of a complete metric space as a basic neighborhood space is unique up to isomorphism.

In BISH the following holds:

(iii) If (X, d) and $(\mathcal{V}, \mathcal{T}_{\#})$ are as above in the theorem, then we can define a metric d' on $(\mathcal{V}, \mathcal{T}_{\#})$ (see def. 4.0.0) by defining $d'(x, y) = d(h(x), h(y))$ for $x, y \in \mathcal{V}$ and h a homeomorphism from $(\mathcal{V}, \mathcal{T}_{\#})$ to (X, d). This metric can be obtained as a morphism from $(\mathcal{V} \times \mathcal{V}, \mathcal{T}_{\#})$ to $\mathbb{R}_{\mathrm{nat}}$ by the construction of $(\mathcal{V}, \mathcal{T}_{\#})$. We then see that the apartness topology and the metric d'-topology coincide, in other words $(\mathcal{V}, \mathcal{T}_{\#})$ is metrizable. We conclude: on a well-chosen basic-neighborhood natural representation of a complete metric space, the metric topology coincides with the apartness topology.

REMARK:

(i) The construction in the proof sketch above merits a closer look, since we do not simply choose each 'rational sphere' $B(a_n, q), q \in \mathbb{Q}$ to be a basic dot. Yet for $\mathbb{R}$ and its corresponding natural space $\mathbb{R}_{\mathrm{nat}}$, choosing all closed rational intervals works fine. We cannot guarantee in the general case (X, d) however, that by taking $V = \{B(a_n, q) | q \in \mathbb{Q}, n \in \mathbb{N}\}$ we end up with a natural space $(\mathcal{V}, \mathcal{T}_{\#})$ which is homeomorphic to (X, d). We do know that for $X = \mathbb{C}_p$, taking $V = \{B(a_n, q) | q \in \mathbb{Q}, n \in \mathbb{N}\}$ gives us a $(\mathcal{V}, \mathcal{T}_{\#})$ which contains 'more' points than $\mathbb{C}_p$. It might prove a nice challenging exercise to the reader to see why this is the case. In the appendix A.2.1 we detail this nice example of a non-archimedean metric natural space.

(ii) For most applied-computational purposes, a basic neighborhood representation of a complete metric space seems the best option. We believe that for $\mathbb{R}$, the representation $\sigma_{\mathbb{R}}$ which we define in the following chapter is a good choice for computational purposes also. Our definition of $\sigma_{\mathbb{R}}$ and $\preceq$-morphisms matches the recommendations in [BauKav2009] for efficient exact real arithmetic.

(iii) That the metric topology coincides with the apartness topology on (a well-chosen basic-neighborhood representation of) a complete metric space, allows for simplification later on. Our definition of 'direct limit' leading to e.g. Silva spaces uses only the apartness topology.

(END OF REMARK)

1.2.4 Metrizability of natural spaces From intuitionistic topology, we can retrieve results on the metrizability of natural spaces. With a definition of the notion ‘star-finitary’ which closely resembles the notion ‘strongly paracompact’, we obtain the constructive theorem that every star-finitary natural space is metrizable.

Also, we can easily define natural spaces which are non-metrizable. Comparable to ideas from Urysohn ([Ury1925a]), in intuitionistic topology one finds spaces with separation properties ‘T_1 but not T_2’ and ‘T_2 but not T_3’ (see [Waa1996]). These spaces can be transposed directly to our setting.

However, a different class of non-metrizable natural spaces arises when we look at direct limits in infinite-dimensional topology. As an example we show that the space of ‘eventually vanishing real sequences’ (which is the direct limit of the Euclidean spaces $(\mathbb{R}^n)_{n\in\mathbb{N}}$) can be formed as a non-metrizable natural space.

We postpone the definitions and theorems to chapter four.

1.2.5 Infinite products are natural Another way to see that the class of natural spaces is large is to look at (in)finite products of natural spaces. We postpone the definitions to paragraph 3.5.1, because of some technical concerns and extra issues. The basic idea to arrive at the natural product of a) a finite sequence b) an infinite sequence of natural spaces is simple however. For (weak) basic neighborhood spaces the (in)finite natural product space is homeomorphic to the Tychonoff product-topology space.[14]

1.3 APPLIED MATH INTERMEZZO: HAWK-EYE, BINARY AND DECIMAL REALS

1.3.0 Hawk-Eye Already we have introduced enough material to discuss an interesting application of mathematics, in the world of professional tennis. In 2006 the multicamera-fed decision-support system Hawk-Eye was first officially used to give players an opportunity to correct erroneous in/out calls. Hawk-Eye uses ball-trajectory data from several precision cameras to calculate whether a given ball was IN: ‘inside the line or touching the line’ or OUT: ‘outside the line’. Hawk-Eye is now widely accepted, for decisions which can value at over $100,000.

[14] See 3.5.2. Products of (weak) basic neighborhood spaces are ‘faithful’. In CLASS and INT also products of ‘star-finitary’ spaces are faithful. We know of no unfaithful products.

The measurements of the cameras can be seen as the 'dots' or 'specks' that we used for illustration in our introduction. Software of Hawk-Eye must in some way run on these dots. The interesting thing is that Hawk-Eye does not have the feature of a LET: 'perhaps the ball was in, perhaps the ball was out, so replay the point'. From this and our work so far we immediately derive:

claim Hawk-Eye, irrespective of the precision of the cameras, will systematically call OUT certain balls which are measurably IN or vice versa.

The claim is not per se important for tennis. Hawk-Eye admits to an inaccuracy of 2-3 mm, and under this carpet the above claim can be conveniently swept (still, one sees 'sure' decisions where the margin is smaller). Hawk-Eye's inaccuracy is usually blamed on inaccuracy of the camera system. But regardless of camera precision we cannot expect to solve the topological problem that there is no natural morphism from the real numbers to a two-point natural space $\{\mathrm{IN}, \mathrm{OUT}\}$ which takes both values IN and OUT. And our recommendation to Hawk-Eye is to introduce a LET feature, see appendix A.2.0 for a more detailed description.

1.3.1 Binary, ternary and decimal real numbers Mathematically more challenging than morphisms from $\mathbb{R}_{\mathrm{nat}}$ to a two-point space are morphisms from $\mathbb{R}_{\mathrm{nat}}$ to the (natural) binary real numbers $\mathbb{R}_{\mathrm{bin}}$ and decimal real numbers $\mathbb{R}_{\mathrm{dec}}$. These morphisms reveal the topology behind different representations of the real numbers on a computer, and transitions between these representations. For simplicity we discuss mainly $\mathbb{R}_{\mathrm{bin}}$, since the situation with $\mathbb{R}_{\mathrm{dec}}$ is completely similar. For some purposes also the ternary real numbers $\mathbb{R}_{\mathrm{ter}}$ come in handy.

DEFINITION: $\mathrm{R}_{\mathbb{Q},\mathrm{bin}} \underset{D}{=} \{\bigcirc_{\mathbb{R}}\} \cup \{[\frac{n}{2^m}, \frac{n+1}{2^m}] \mid n \in \mathbb{Z}, m \in \mathbb{N}\}$, $\mathrm{R}_{\mathbb{Q},\mathrm{ter}} \underset{D}{=} \{\bigcirc_{\mathbb{R}}\} \cup \{[\frac{n}{3^m}, \frac{n+1}{3^m}] \mid n \in \mathbb{Z}, m \in \mathbb{N}\}$ and $\mathrm{R}_{\mathbb{Q},\mathrm{dec}} \underset{D}{=} \{\bigcirc_{\mathbb{R}}\} \cup [\frac{n}{10^m}, \frac{n+1}{10^m}] \mid n \in \mathbb{Z}, m \in \mathbb{N}\}$.
Then $\mathbb{R}_{\mathrm{bin}} = (\mathrm{R}_{\mathbb{Q},\mathrm{bin}}, \#_{\mathbb{R}}, \preceq_{\mathbb{R}})$ is the natural space of the *binary real numbers*. Similarly we form the corresponding natural spaces $\mathbb{R}_{\mathrm{ter}}$ and $\mathbb{R}_{\mathrm{dec}}$ of the *ternary* and *decimal real numbers*.

Put $[0,1]_{\mathrm{R,bin}} \underset{D}{=} \{[\frac{n}{2^m}, \frac{n+1}{2^m}] \mid n, m \in \mathbb{N} \mid n < 2^m\}$, $[0,1]_{\mathrm{R,ter}} \underset{D}{=} \{[\frac{n}{3^m}, \frac{n+1}{3^m}] \mid n, m \in \mathbb{N} \mid n < 3^m\}$ and $[0,1]_{\mathrm{R,dec}} \underset{D}{=} \{[\frac{n}{10^m}, \frac{n+1}{10^m}] \mid n, m \in \mathbb{N} \mid n < 10^m\}$ to form the corresponding natural spaces $[0,1]_{\mathrm{bin}}$, $[0,1]_{\mathrm{ter}}$ and $[0,1]_{\mathrm{dec}}$, each with the same maximal dot $[0,1]$ denoted by $\bigcirc_{[0,1]}$.

Notice that as a partial order, $(\mathrm{R}_{\mathbb{Q},\mathrm{bin}}, \preceq_{\mathbb{R}})$ is a tree. The reader can think of the natural binary reals as the set of those real numbers x that can also

be given as a *binary expansion* $x = (-1)^s \cdot \Sigma_{n\in\mathbb{N}} a_n \cdot 2^{-n+m}$, where $s \in \{0, 1\}$, $m \in \mathbb{N}$ and $a_n \in \{0, 1\}$ for all $n \in \mathbb{N}$, such that $m > 0$ implies $a_0 \neq 0$. We call s the *sign* and write $s = +, -$ for $s = 0, 1$ respectively. We call m the *binary point place*. Then the $(a_n)_{n\in\mathbb{N}}$ are the *binary digits* in this binary expansion of x, and we write $x = (s)\, a_0\, a_1 \ldots a_m . a_{m+1} \ldots$. Notice the *binary point* that we write between a_m and a_{m+1} to denote the binary point place.
Replacing 'binary, 2' with 'ternary, 3' and 'decimal, 10' respectively, we obtain the similar definitions for $\mathbb{R}_{\text{ter}}$ and $\mathbb{R}_{\text{dec}}$. (END OF DEFINITION)

Classically every real number y has an equivalent binary expansion, but in computational practice and in constructive mathematics this is not the case (see e.g. [GNSW2007] for a thorough discussion). So with $\mathbb{R}_{\text{bin}}$, $\mathbb{R}_{\text{ter}}$ and $\mathbb{R}_{\text{dec}}$ we in practice obtain *different* representations of the real numbers. We wish to shed some light on the natural topology involved in the (im)possible transition from one such representation to another.

1.3.2 Morphisms to and from the binary reals

It turns out that a morphism f from $\mathbb{R}_{\text{nat}}$ to $\mathbb{R}_{\text{bin}}$ which is order preserving ($x \leq_{\mathbb{R}} y$ implies $f(x) \leq_{\mathbb{R}} f(y)$) has to be 'locally constant' around the f-originals of the rationals $\{\frac{k}{2^m} \mid k, m \in \mathbb{N}\}$. These are the points where the binary expansion has two alternatives (e.g. for 1 both $0.111\ldots \equiv 0 + 1\cdot 2^{-1} + 1\cdot 2^{-2} + 1\cdot 2^{-3} + \ldots$ and $1.000\ldots \equiv 1 + 0\cdot 2^{-1} + 0\cdot 2^{-2} + 0\cdot 2^{-3} + \ldots$ are binary representations). Since these binary rational numbers lie dense in $\mathbb{R}$, there can be no injective morphism from $\mathbb{R}_{\text{nat}}$ to $\mathbb{R}_{\text{bin}}$ (notice that any injective morphism f from $\mathbb{R}_{\text{nat}}$ to $\mathbb{R}_{\text{nat}}$ is either order preserving, or order reversing in which case a similar argument for local constancy obtains). But this does not mean that all morphisms from $\mathbb{R}_{\text{nat}}$ to $\mathbb{R}_{\text{bin}}$ are constant. We will derive a non-constant morphism from $\mathbb{R}_{\text{nat}}$ to $\mathbb{R}_{\text{ter}}$ from our metrization theorem in 4.0.8. We can easily turn this into a non-constant morphism from $[0, 1]$ to $[0, 1]_{\text{bin}}$, when we realize that $[0, 1]_{\text{bin}}$ and $[0, 1]_{\text{ter}}$ are isomorphic.

The well-known *Cantor function* f_{Can} (also known as 'the devil's staircase') is another example of a non-constant natural morphism from $[0, 1]$ to $[0, 1]_{\text{bin}}$. The Cantor function is most easily described as a refinement morphism from $[0, 1]_{\text{ter}}$ to $[0, 1]_{\text{bin}}$, but also can be given as a trail morphism on $[0, 1]$.[15]

We now have an example in CLASS of a continuous function between natural spaces which cannot be represented by a morphism. In CLASS, the identity is a homeomorphism from $\mathbb{R}_{\text{nat}}$ to $\mathbb{R}_{\text{bin}}$ (remember that in CLASS we work with

[15] We leave this latter statement as a non-trivial exercise to the reader though, see A.2.2.

the equivalence classes, and that every real number has an equivalent binary representation). But this identity cannot be represented by a natural morphism, as we pointed out above. In the light of theorem 1.2.2, the 'reason' for this is that $\mathbb{R}_{\mathrm{bin}}$ is not a basic neighborhood space, which we can easily verify by looking at the real number $\frac{1}{2}$. In fact, in $\mathbb{R}_{\mathrm{bin}}$, of the basic dots only the maximal dot is a neighborhood of $\frac{1}{2}$.

We will make the above statements and definitions precise in the appendix A.2.2, also showing the equivalence between the reals allowing a binary expansion and the binary reals. We then use the ternary reals to construct the Cantor set $\mathcal{C}_{[0,1]}$, and the ContraCantor set, which is a compact subspace $\text{Ↄ}_{[0,1]}$ of $[0,1]$ such that: $d_{\mathbb{R}}(\text{Ↄ}_{[0,1]}, \mathcal{C}_{[0,1]}) = 0$ and yet $d(x,y) > 0$ for all recursive $x \in \text{Ↄ}_{[0,1]}, y \in \mathcal{C}_{[0,1]}$. So in RUSS we have $d_{\mathbb{R}}(x, \mathcal{C}_{[0,1]}) > 0$ for $\forall x \in \text{Ↄ}_{[0,1]}$, giving us a RUSS example of two disjoint complete compact spaces with distance 0.

REMARK: One can show with little effort that for $n, m \in \mathbb{N}$ the n-ary and m-ary reals are $\preceq$-isomorphic. However, we believe the n-ary reals can only be identically embedded in the m-ary reals if there is a $b \geq 1$ in $\mathbb{N}$ such that m divides n^b (for an identical embedding f we have $f(x) \equiv_{\mathbb{R}} x$ for all x). This gives a natural-topological classification of the different n-ary representations of real numbers. We leave this as an exercise. (END OF REMARK)

1.4 INTUITIONISTIC PHENOMENA IN NATURAL TOPOLOGY

1.4.0 Pathwise and arcwise connectedness

From the previous paragraph we deduce an interesting property of $\mathbb{R}_{\mathrm{bin}}$ (and $\mathbb{R}_{\mathrm{ter}}$, $\mathbb{R}_{\mathrm{dec}}$): it is a pathwise$_{\mathrm{nat}}$ connected space which is not arcwise$_{\mathrm{nat}}$ connected. For this we must define:

DEFINITION: A natural space $(\mathcal{V}, \mathcal{T}_{\#})$ is called *pathwise*$_{\mathrm{nat}}$ (resp. *arcwise*$_{\mathrm{nat}}$) *connected* iff for all $x, y \in V$ there is a morphism (resp. an injective morphism) f from $[0,1]_{\mathrm{nat}}$ to $(\mathcal{V}, \mathcal{T}_{\#})$ such that $f(0) \equiv x$ and $f(1) \equiv y$. (END OF DEFINITION)

THEOREM: $\mathbb{R}_{\mathrm{bin}}$ (as well as $\mathbb{R}_{\mathrm{ter}}$, $\mathbb{R}_{\mathrm{dec}}$) is a pathwise$_{\mathrm{nat}}$ connected space which is not arcwise$_{\mathrm{nat}}$ connected.

PROOF: A detailed constructive proof for $\mathbb{R}_{\text{ter}}$ is given in [Waa1996] in an intuitionistic setting. We indicate the translation to our setting in the appendix A.3.3. The reader should have no trouble giving a proof using the above subsections. (END OF PROOF)

1.4.1 Intuitionistic phenomena arise naturally The previous examples and theorems are a quite faithful mirror of the phenomena studied in intuitionism (INT). In INT, an elegant class of natural spaces (called 'spreads') is studied. Hereby the main intuitionistic axiom which is not classically valid is the continuity principle **CP** (and by implication its stronger versions $\mathbf{AC}_{10}$ and $\mathbf{AC}_{11}$, see the appendix A.4). A main consequence of **CP** is that total functions on spreads are always given by morphisms. In natural topology, by directly considering morphisms we create a simple classical mirror of many intuitionistic results. Not surprisingly, these results have direct computational meaning, and are therefore also of relevance for applied mathematics.

Let's introduce **CP** here, to see what sets INT apart from CLASS axiomatically.[16] **CP** is formulated for Baire space $\mathbb{N}^{\mathbb{N}}$, we show that Baire space is a universal natural space in the next section. For an element α of Baire space $\mathbb{N}^{\mathbb{N}}$ we write $\overline{\alpha}(n)$ to denote the finite sequence formed by the first n values of α.

$\boxed{\mathbf{CP}}$ Let A be a subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ such that $\forall \alpha {\in} \mathbb{N}^{\mathbb{N}} \exists n {\in} \mathbb{N}\, [(\alpha, n) {\in} A]$. Then $\forall \alpha {\in} \mathbb{N}^{\mathbb{N}} \exists m, n {\in} \mathbb{N}\ \forall \beta {\in} \mathbb{N}^{\mathbb{N}}\, [\overline{\alpha}(m) {=} \overline{\beta}(m) \to (\beta, n) {\in} A]$.

The motivation for this axiom is that in INT infinite sequences arise step-by-step, and that among these sequences are also those about which we know -at any given time m- nothing more than the first m values. We can also form recursive determinate sequences, no problem, but asserting that for *all* α one can find an n such that $(\alpha, n) {\in} A$ means that the indeterminate sequences are included. For any such indeterminate sequence α one has to produce the favourable n with $(\alpha, n) {\in} A$ at some point in time, say m. At this point, we know only the first m values of α, which implies that for $\beta {\in} \mathbb{N}^{\mathbb{N}}$ with $\overline{\beta}(m) {=} \overline{\alpha}(m)$, we also have $(\beta, n) {\in} A$.

The other intuitionistic axioms (apart from the strengthenings of **CP**) are all valid classically, and the above axiom also even makes sense classically in

[16] Of course, apart from axioms there is also a fundamental conceptual difference between constructive mathematics and classical mathematics, regarding infinity and omniscience, see also 4.2.1.

the right setting.[17] We think that such a setting is obtained naturally when considering a two-player game with limited information, see also section 4.2.4). In this setting we can even prove **CP**. Notice that this setting strongly resembles that of our engineer taking measurements from nature. This gives a philosophical explanation for the aptness of INT for physics.

However, by considering morphisms on natural spaces a classical mathematician can skip most of this issue and still see intuitionistic phenomena arising naturally and quite faithfully.

We have come to the end of this chapter. Its main purpose, apart from giving the basic definitions and properties, is to give some inkling of the relation between constructive topology and applied mathematics (our APPLIED perspective). In the next chapters we will turn predominantly to the other perspectives GENERAL, CONSTRUCTIVE and PHYSICS.

[17] The author considers the axiom to be simply true in its intended context. He thinks intuitionistic mathematics deserves a prime role in mathematical investigations.

CHAPTER TWO

Baire space is universal

Natural Baire space is a universal natural space, meaning that every natural space is the image of Baire space under a natural morphism.

Its set of basic opens can be pictured as a tree, and this partial-order property provides a fundamental simplification. For many natural spaces, the related concept of 'trea' is similarly useful.

Through Baire space, a direct link with intuitionism (INT) can be established. We define natural spreads and spraids (corresponding to trees and treas) as well as fans and fanns. This enables us to show, for the APPLIED perspective, that continuous $\mathbb{R}$-to-$\mathbb{R}$-functions can be represented by computationally efficient morphisms.

Cantor space is a universal fan. Compactness of Cantor space is seen to depend on the axiom **FT** (Fan Theorem), derived from **BT** (Brouwer's Thesis), which holds in CLASS and INT but not in RUSS. In RUSS, Baire space is isomorphic to Cantor space.

Since we wish to work in BISH, we do not adopt **BT**, and turn to inductive definitions instead.

2.0 INTRODUCTION TO BAIRE SPACE

2.0.0 **Classical Baire space** In CLASS, Baire space is $\mathbb{N}^{\mathbb{N}}$ with the usual product topology. Baire space is a universal Polish space ('Polish' meaning 'second countable completely metrizable'), that is: every Polish space is the continuous image of Baire space.[18] However, in fact Baire space is universal for a certain larger class of spaces. We can see this classically by looking at any equivalence relation $\equiv$ on $\mathbb{N}^{\mathbb{N}}$, and forming the corresponding quotient topology and quotient space. In this way we trivially see that Baire space is universal for the class of topological spaces homeomorphic to a quotient space of Baire space. In this paper we study this larger class from our natural perspective. For constructive reasons we limit ourselves to quotient topologies derived from a Π^1_0-equivalence relation. To see that this class is larger than the class of Polish spaces, it suffices to see that some of these quotient spaces are non-metrizable.

2.0.1 **Introduction to natural Baire space** One goal of this paper is to give a classical mathematician (our perspective GENERAL) an understanding of what constructive and intuitionistic topology 'is all about', in such a way that little foundational terminology is necessary. We believe that the (classically valid) setting of natural spaces and natural morphisms between them, is a faithful mirror of Brouwer's basic intuitionistic setting of 'spreads' and 'spread-functions'.

To see that our class of natural spaces is in fact quite large, we will show that it encompasses (representations of) every Polish space. Also, (in)finite products of natural spaces are representable as a natural space. Another result already noted by Brouwer is that natural Baire space $\mathcal{N} = \mathbb{N}^{\mathbb{N}}_{\mathrm{nat}}$ is a universal natural space, meaning that every natural space $(\mathcal{V}, \mathcal{T}_{\#})$ is the image of natural Baire space under some natural morphism from $\mathcal{N}$ to $(\mathcal{V}, \mathcal{T}_{\#})$.

To put it differently, from the GENERAL perspective we study (subspaces of) Baire space endowed with a quotient topology derived from a Π^1_0-equivalence relation. BUT: our view is finer than the usual classical topological view. Using morphisms we can distinguish between interesting spaces that are classically homeomorphic (indistinguishable with topological methods). AND: our view and constructive methods can be fruitful in applied math and physics.

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[18] This can often be used to streamline proofs for separable complete metric spaces.

Quite some work has already been done in intuitionistic topology, with Baire space as fundament.[19] We will be able to retrieve some nice results (e.g. regarding metrizability of natural spaces) in a simple way, once we have shown that our setting of natural spaces is indeed a faithful mirror of Brouwer's setting. Since we take a neutral constructive approach, we will not use any specific classical or intuitionistic axioms. However we will freely use the axioms of countable choice $\mathbf{AC}_{01}$ and dependent choice $\mathbf{DC}_1$, which are generally accepted as constructive (also see A.4).

2.0.2 Natural Baire space There is much mathematical beauty in Baire space, and its definition as natural space is likewise elegant. Its set of basic dots is $\mathbb{N}^*$, the set of all finite sequences of natural numbers (representing the basic clopen sets of Baire space). We use the definition also to relate natural Baire space to usual Baire space.

DEFINITION: Let $\mathbb{N}^*$ be the set of all finite sequences of natural numbers. For $a = a_0, \ldots, a_i$, $b = b_0, \ldots, b_j \in \mathbb{N}^*$ the concatenation $a_0, \ldots, a_i, b_0 \ldots, b_j$ is denoted by $a \star b$. Define: $b \preceq_\omega a$ iff there is c such that $b = a \star c$. Define: $a \#_\omega b$ iff $a \npreceq_\omega b$ and $b \npreceq_\omega a$.

Then $(\mathbb{N}^*, \#_\omega, \preceq_\omega)$ is a pre-natural space, with the empty sequence as maximal dot, which we also denote $\bigcirc_\omega$ or simply $\bigcirc$. Its corresponding natural space $(\mathcal{N}, \mathcal{T}_{\#_\omega})$ we call *natural Baire space*. We also write $\mathbb{N}^{\mathbb{N}}_{\text{nat}}$ for $\mathcal{N}$.

In addition, let $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $m \in \mathbb{N}$, then we write $\overline{\alpha}(m)$ for the finite sequence $\alpha(0), \ldots, \alpha(m-1)$ consisting of the first m values of α. Notice that $\overline{\alpha}(m)$ is an element of $\mathbb{N}^*$, so the sequence $\overline{\alpha} = \overline{\alpha}(0), \overline{\alpha}(1), \ldots$ is a point in $\mathcal{N}$.
Conversely, for a point $p \in \mathcal{N}$, there is a unique sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $p \equiv_\omega \overline{\alpha}$. We write p^* for this unique α, giving that $p \equiv_\omega \overline{p^*}$ for $p \in \mathcal{N}$ and $\alpha = \overline{\alpha}^*$ for $\alpha \in \mathbb{N}^{\mathbb{N}}$.[20] (END OF DEFINITION)

THEOREM: $(\mathcal{N}, \mathcal{T}_{\#_\omega})$ is homeomorphic with $(\mathbb{N}^{\mathbb{N}}, \mathcal{T}_{\text{prod}})$.

PROOF: We leave it to the reader to verify that the function $\alpha \to \overline{\alpha}$ from $\mathbb{N}^{\mathbb{N}}$ to $\mathcal{N}$ defined above is a homeomorphism, with inverse $p \to p^*$ (also defined above). (END OF PROOF)

[19] Brouwer already gave an intuitionistic proof of the Jordan curve theorem, to mention just one historic result.

[20] This also somewhat relates to the axiom of extensionality, see A.4.7.

2.0.3 **Natural Cantor space** We first define the notion 'natural subspace', since in natural Cantor space we have a prime example.

DEFINITION: Let $(\mathcal{V}, \mathcal{T}_{\#})$ be a natural space derived from $(V, \#, \preceq)$. Let W be a countable subset of V, then $(W, \#, \preceq)$ is a pre-natural space, with corresponding set of points $\mathcal{W}$. If $(\mathcal{W}, \mathcal{T}_{\#})$ is a natural space (see def. 1.0.6), then we call $(\mathcal{W}, \mathcal{T}_{\#})$ a *natural subspace* of $(\mathcal{V}, \mathcal{T}_{\#})$ iff in addition $(\mathcal{W}, \mathcal{T}_{\#})$ as a natural space coincides with $(\mathcal{W}, \mathcal{T}_{\#})$ as a topological subspace of $(\mathcal{V}, \mathcal{T}_{\#})$ (in the subspace topology '$U \subseteq \mathcal{W}$ is open' is defined thus: there is an open $U' \subseteq \mathcal{V}$ such that $U = U' \cap \mathcal{W}$).
Let $\{0,1\}^*$ be the set of finite sequences of elements of $\{0,1\}$. Now *natural Cantor space* is the natural subspace $(\mathcal{C}, \mathcal{T}_{\text{nat}})$ of natural Baire space formed by the pre-natural space $(\{0,1\}^*, \#_\omega, \preceq_\omega)$ and its set of points $\mathcal{C}$. (END OF DEFINITION)

REMARK: We will see that any natural subspace of natural Baire space is the image of Baire space under a (continuous) morphism. From descriptive set theory, it follows that many topological subspaces of a natural space cannot be represented as a natural subspace. We will give an example of such a space in 2.4.2. The notion of 'natural subspace' is weaker than the intuitionistic notion of 'subspread'. We will define this notion in our context also. Natural Cantor space is homeomorphic to usual Cantor space, and corresponds directly to Brouwer's fan σ_2. (END OF REMARK)

From now on, when the context is clear we will simply say 'Baire space' and 'Cantor space' and omit the extra word 'natural'.

2.1 LATTICES, TREES AND SPREADS

2.1.0 **Lattices and posets of basic dots** In topology, the open sets form a lattice structure under the inclusion relation. This structure is often exploited in various ways. One way is to (somewhat) disregard meet and join operations and focus simply on the partial-order properties (of the 'poset' of opens). Our basic dots in general need not form a lattice, but their partial-order properties play an important role. This role could even be too restrictive, without trail morphisms.

We now go into these partial-order properties in more detail. This requires some attention from the reader, but the rewards are great. Eventually we will derive from these properties some important simplifications which enable us to forge a direct link with intuitionistic topology. Brouwer's simple and elegant concept of a spread then becomes accessible to us as well.

2.1.1 **Trees and treas** The elegance of Baire space can be seen as stemming from the fact that its poset of basic dots $(\mathbb{N}^*, \preceq_\omega)$ forms a countable tree. That is: for any dot $a = a_0, \ldots, a_{n-1} \in \mathbb{N}^*$, there is a *unique* finite trail of immediate successors/predecessors from $\bigcirc_\omega$ to a. (Therefore any $\prec$-trail between dots is finite, and also the successor/predecessor relationship is decidable.). We cannot hope to achieve this elegance for any natural space, but we can show that any natural space $(\mathcal{V}, \mathcal{T}_{\#_1})$ is isomorphic to a natural space $(\mathcal{W}, \mathcal{T}_{\#_2})$ where $(W, \preceq_2)$ equals $(\mathbb{N}^*, \preceq_\omega)$. Or more practical: where $(W, \preceq_2)$ is a full subtree of $(\mathbb{N}^*, \preceq_\omega)$, definition follows.

This means that we could limit ourselves to natural spaces $(\mathcal{V}, \mathcal{T}_{\#})$ where $(V, \preceq)$ is (a full subtree of) $(\mathbb{N}^*, \preceq_\omega)$. This has a strong simplifying effect, which gives much beauty to Brouwer's intuitionism. The only downside is that for many natural spaces, we have to replace our original basic dots with elements of $\mathbb{N}^*$, which can in practice be a tedious encoding. Therefore we propose the compromise notion of a 'trea'. One can think of a trea as being a tree wherein certain of the branches have been neatly glued together in a number of places. A more precise characterization of a trea: a countable $\prec$-directed acyclic graph with a maximal element, where for each node there are finitely many immediate-predecessor trails to the maximal element, all of the same length. Another characterization: a countable $\preceq$-poset with a maximal element where each point has finitely many immediate-predecessor trails to the maximal element, all of the same length.[21]

DEFINITION: Let $(\mathcal{V}, \mathcal{T}_{\#})$ be a natural space, with corresponding $(V, \#, \preceq)$, and let $(\mathcal{W}, \mathcal{T}_{\#})$ with corresponding $(W, \#, \preceq)$ be a natural subspace of $(\mathcal{V}, \mathcal{T}_{\#})$ (so $W \subseteq V$). Let $a \prec c$ in V.

(i) We say that a is a *successor* of c in $(V, \preceq)$ (notation $a \propto_V c$, or simply $a \propto c$ if the context is clear) iff for all $b \in V$, if $a \prec b \preceq c$ then $b = c$. A sequence $b_0 \propto \ldots \propto b_n$ in V is called a *$\propto$-trail of length n from b_0 to b_n in* $(V, \preceq)$. For $b \in V$ we put $\propto_V(b) =_D \{d \in V \mid d \propto b\}$, and simply write $\propto(b)$ when the context is clear.

[21] We apologize for the lengthy definitions. The concepts are not too difficult, and will provide elegant simplification later on.

(ii) $(V, \preceq)$ is called a *trea* iff for every $a \in V$ the set $\{b \in V \mid a \preceq b\}$ of predecessors of a is finite (then the successor relation $\propto$ is decidable, and for every $a \in V$ there are finitely many $\propto$-trails from $\bigcirc$ to a) and in addition there is an integer $\lg(a) \in \mathbb{N}$ such that every $\propto$-trail from $\bigcirc$ to a has length $\lg(a)$.[22]

(iii) Now let $(V, \preceq)$ be arbitrary, where $(W, \preceq)$ is a tree (trea), then we say that $(W, \preceq)$ is a *subtree (subtrea)* of $(V, \preceq)$.

(iv) Let $(V, \preceq)$ be a tree (trea), and $(W, \preceq)$ a subtree (subtrea). We then call $(W, \preceq)$ a *full subtree (subtrea)* of $(V, \preceq)$ iff $b \propto_W d$ implies $b \propto_V d$ for all $b, d \in W$. (Then each $\propto_W$-trail in $(W, \preceq)$ is a $\propto_V$-trail in $(V, \preceq)$).

(END OF DEFINITION)

As stated above, we can show that any natural space $(\mathcal{V}, \mathcal{T}_{\#_1})$ is isomorphic to a natural space $(\mathcal{W}, \mathcal{T}_{\#_2})$ where $(W, \preceq_2)$ is a full subtree of $(\mathbb{N}^*, \preceq_\omega)$. In intuitionism therefore, most notions are defined for full subtrees of $(\mathbb{N}^*, \preceq_\omega)$. But since we wish to incorporate the possibility to avoid encoding schemes, we will define our notions for treas and full subtreas. Treas behave just like trees (and any tree is a trea). Most of the spaces of interest that we mentioned so far (see 1.2.1) have an intuitive representation as a natural space $(\mathcal{V}, \mathcal{T}_\#)$ where $(V, \preceq)$ is a trea. One reason for this is that the infinite product of a sequence of treas can be naturally built as a trea.[23] One might find another reason in the existence of star-finite refinements of per-enumerable open covers of metric spaces, see [Waa1996].

EXAMPLE: For the natural real numbers $\mathbb{R}_{nat}$ we can easily indicate an isomorphic subspace $(\sigma_\mathbb{R}, \mathcal{T}_{\#_\mathbb{R}})$ with corresponding pre-natural space $(\sigma_\mathbb{R}, \#_\mathbb{R}, \preceq_\mathbb{R})$, where $(\sigma_\mathbb{R}, \preceq_\mathbb{R})$ is a trea:

$$(\sigma_\mathbb{R}, \#_\mathbb{R}, \preceq_\mathbb{R}) \underset{D}{=} (\{\bigcirc_\mathbb{R}\} \cup \{[\tfrac{n}{2^m}, \tfrac{n+2}{2^m}] \mid n \in \mathbb{Z}, m \in \mathbb{N}\}, \#_\mathbb{R}, \preceq_\mathbb{R}).$$

Our examples in 1.3.1 should show why we cannot hope to find an isomorphic subspace $(\mathcal{V}, \mathcal{T}_{\#_\mathbb{R}})$ where $(V, \preceq)$ is a tree (!).

2.1.2 Spreads and spraids

The previous example illuminates a bridge towards intuitionistic terminology, which we give in the following definition:

[22] This last condition more or less follows from the first. If we only stipulate that for every $a \in V$ the set $\{b \in V \mid a \preceq b\}$ is finite, then we can add extra basic dots to V to end up with an isomorphic space in which all $\propto_V$-trails from $\bigcirc$ to a given a have the same length.

[23] Using finite products of basic dots from the treas involved, see 3.5.1.

DEFINITION: Let $(\mathcal{V}, \mathcal{T}_\#)$ be a natural space, with corresponding $(V, \#, \preceq)$, and let $(\mathcal{W}, \mathcal{T}_\#)$ with corresponding $(W, \#, \preceq)$ be a *decidable* natural subspace of $(\mathcal{V}, \mathcal{T}_\#)$ (meaning W is a decidable subset of V).

(i) We call $(\mathcal{V}, \mathcal{T}_\#)$ a *spread (spraid)* iff $(V, \preceq)$ is a tree (trea) and each infinite $\prec$-trail defines a point (see also A.5.5). Then we call $(\mathcal{W}, \mathcal{T}_\#)$ a *subspread (subspraid)* of $(\mathcal{V}, \mathcal{T}_\#)$ iff $(W, \preceq)$ is a full subtree (subtrea) of $(V, \preceq)$.

(ii) We call $(\mathcal{V}, \mathcal{T}_\#)$ a *Baire spread* iff $(\mathcal{V}, \mathcal{T}_\#)$ is a subspread of Baire space.

(iii) For any space $(\mathcal{V}, \mathcal{T}_\#)$, if $(\mathcal{W}, \mathcal{T}_\#)$ is a spread (spraid), then we simply call $(\mathcal{W}, \mathcal{T}_\#)$ a weak subspread (subspraid) of $(\mathcal{V}, \mathcal{T}_\#)$. (If $(V, \preceq)$ contains an infinite $\prec$-trail between two dots, then we could drop the prefix 'weak'.[24]).

By extension, $(\mathcal{V}, \mathcal{T}_\#)$ is *spreadlike* iff there is an isomorphism between $(\mathcal{V}, \mathcal{T}_\#)$ and a spread. (END OF DEFINITION)

EXAMPLE: Important basic examples of subspraids are obtained as follows. For $(\mathcal{V}, \mathcal{T}_\#)$ a spraid and a in V, one easily sees that $V_a = \{b \in V \mid b \preceq a\} = \{a\}_\preceq$ determines a subspraid of V if we put its maximal dot as $\bigcirc_a = a$. These basic subspraids are important later on in defining 'genetic induction'.

2.2 UNIVERSAL NATURAL SPACES

2.2.0 **Baire space is universal** Baire space is a universal natural space, by which we mean that each natural space can be seen as the image of Baire space under a natural morphism. Brouwer already realized this, and simplified his concepts accordingly.[25] In our setting, we adopt a similar simplification, for esthetic reasons and to save paper and energy. We show that every natural space is spreadlike, which on a meta-level gives us a one-on-one correspondence with many important intuitionistic results.

THEOREM: Every natural space is spreadlike. In fact, every natural space $(\mathcal{V}, \mathcal{T}_\#)$ is isomorphic to a spread $(\mathcal{W}, \mathcal{T}_\#)$ whose tree is $(\mathbb{N}^*, \preceq_\omega)$.

[24] If $(V, \preceq)$ contains such an infinite trail, then $(V, \preceq)$ is not a trea and $(\mathcal{V}, \mathcal{T}_\#)$ is not a spraid. So the condition for spraids and subspraids, that $(W, \preceq)$ is a full subtrea of $(V, \preceq)$, becomes void. However we cannot always decide whether $(V, \preceq)$ is a trea or not.

[25] Often in difficult language...

COROLLARY:

(i) Let $(\mathcal{V}, \mathcal{T}_\#)$ be a natural space, then there is a surjective $\preceq$-morphism from Baire space to $(\mathcal{V}, \mathcal{T}_\#)$. (‘Baire space is a universal spread’, ‘every natural space is the natural image of Baire space’, ‘every natural space is a quotient topology of Baire space’).

(ii) If $(\mathcal{V}, \mathcal{T}_\#)$ is a basic-open space (see definition 1.2.2) then $(\mathcal{V}, \mathcal{T}_\#)$ is isomorphic to a basic-open spread $(\mathcal{W}, \mathcal{T}_\#)$ whose tree is $(\mathbb{N}^*, \preceq_\omega)$.

PROOF: Not trivial, see A.3.4 where we give a self-contained proof. (END OF PROOF)

From the theorem we reobtain Brouwer’s simplification: *without any loss of generality we may assume that a given natural space $(\mathcal{V}, \mathcal{T}_\#)$ is a spread*. Points in $\mathcal{V}$ can be constructed step-by-step, as infinite trails in the countable tree $(V, \preceq)$. Any such tree can of course be embedded as a full subtree in $(\mathbb{N}^*, \preceq_\omega)$.

The corollary gives the equivalent picture that each natural space $(\mathcal{V}, \mathcal{T}_\#)$ with corresponding pre-natural space $(V, \#, \preceq)$ is in fact nothing but a pre-apartness $\#_V$ and a refinement relation $\preceq_V$ on $\mathbb{N}^*$ which respect $\#_\omega$ and $\preceq_\omega$. To define these decidable relations we only have to ‘pull back’ the decidable relations $\#$ and $\preceq$ using the given surjective morphism f thus: for $a, b \in \mathbb{N}^*$ put $a \#_V b$ resp. $a \preceq_V b$ iff $f(a) \# f(b)$ resp. $f(a) \preceq f(b)$. Then $a \#_V b$ resp. $a \preceq_\omega b$ implies $a \#_\omega b$ resp. $a \preceq_V b$.

An equivalent situation which avoids encoding arises whenever a natural space $(\mathcal{V}, \mathcal{T}_\#)$ contains a subspraid on which the identity is an isomorphism. Then from the often vast partial-order universe of $(V, \preceq)$ we can restrict ourselves to a subtrea. We give an important example below where the isomorphic subspace is a spraid. We believe this to be the most common setting for natural spaces. In the uncommon case that we cannot find an isomorphic subspace which is a spraid, we can always invoke Brouwer’s encoding to find an isomorphic spread.

EXAMPLE: Looking at the natural real numbers $\mathbb{R}_{\mathrm{nat}}$, we can easily indicate an isomorphic subspace which is a spraid as in example 2.1.1. Put

$$\sigma_\mathbb{R} \underset{D}{=} \{\bigcirc_\mathbb{R}\} \cup \{[\tfrac{n}{2^m}, \tfrac{n+2}{2^m}] \mid n \in \mathbb{Z}, m \in \mathbb{N}\}.$$

Then $(\sigma_\mathbb{R}, \#_\mathbb{R}, \preceq_\mathbb{R})$ is a spraid which is an isomorphic subspace of $\mathbb{R}_{\mathrm{nat}}$. To turn this spraid into an isomorphic spread, we only have to ‘unglue’. The best way to do so is to look at the trail space $\sigma_{\mathbb{R}}^{\wr}$ of $\sigma_{\mathbb{R}}$ (see def. 1.1.4). Specifically, we look at the $\propto$-trails in $\sigma_{\mathbb{R}}^{\wr}$ which (if not equal to the empty sequence $\bigcirc^*$) start with a basic interval in $\propto(\bigcirc_{\mathbb{R}}) = \{[m, m+2] \mid m \in \mathbb{Z}\}$. So put:

$$\sigma_{\mathbb{R}}^{\between} \underset{D}{=} \{a = a_0, \ldots a_{n-1} \in \sigma_{\mathbb{R}}^{\wr} \mid n \in \mathbb{N} \mid a \text{ is a } \propto\text{-trail and } n \geq 1 \rightarrow \lg(a_0) = 1\}$$

Then $\sigma_{\mathbb{R}}^{\between}$ has as maximal dot $\bigcirc^*$, and an example of a basic dot in $\sigma_{\mathbb{R}}^{\between}$ is the sequence $[0,2], [1,2]$, which has as unglued twin the basic dot $[1,3], [1,2]$. For simplicity, we also write $\sigma_{\mathbb{R}}^{\between}$ for the spread derived from the pre-natural space $(\sigma_{\mathbb{R}}^{\between}, \preceq^*, \#^*)$, which is the unglued version of $\sigma_{\mathbb{R}}$. Similarly we define:

$$\sigma_{[0,1]} \underset{D}{=} \{[\tfrac{n}{2^m}, \tfrac{n+2}{2^m}] \mid n, m \in \mathbb{N} \mid n+2 \leq 2^m, m \geq 1\},$$

so that taking $\bigcirc_{[0,1]} = [0,1]$ we get a subfann $\sigma_{[0,1]}$ of $\sigma_{\mathbb{R}}$ which is isomorphic to $[0,1]$. We can also unglue this subfann, as a subfan $\sigma_{[0,1]}^{\between}$ of $\sigma_{\mathbb{R}}^{\between}$. We leave the details to the reader.

Another more involved example of a spraid arises when building the natural space $C^{\text{unif}}([0,1], \mathbb{R})_{\text{nat}}$ of uniformly continuous functions from $[0,1]$ to $\mathbb{R}$. We will sketch this in the appendix, see A.2.4, referring for details to earlier work of Brouwer.

2.2.1 Cantor space is a universal fan

Similar to Baire space being a universal spread, Cantor space is a universal fan, by which we mean that each 'finitely branching' spraid can be seen as the image of Cantor space under a natural morphism:[26]

DEFINITION: Let $(\mathcal{V}, \mathcal{T}_{\#})$ be a spread (spraid) with corresponding $(V, \#, \preceq)$, so $(V, \preceq)$ is a tree (trea). We call $(V, \preceq)$ *finitely branching* iff for all $c \in V$ the set $\propto(c) = \{a \in V \mid a \propto c\}$ is finite. We call $(\mathcal{V}, \mathcal{T}_{\#})$ a *fan (fann)* iff $(V, \preceq)$ is a finitely branching tree (trea). By extension, $(\mathcal{V}, \mathcal{T}_{\#})$ is *fanlike* iff $(\mathcal{V}, \mathcal{T}_{\#})$ is isomorphic to a fan.[27] (END OF DEFINITION)

THEOREM: Let $(\mathcal{V}, \mathcal{T}_{\#})$ be a fann, then there is a surjective morphism from Cantor space to $(\mathcal{V}, \mathcal{T}_{\#})$. ('Cantor space is a universal fan').

COROLLARY: Every fann is fanlike. Every fanlike space is the natural image of Cantor space.

[26] This is also a result due to Brouwer.

[27] Any fann is fanlike so we need not define 'fannlike'.

PROOF: See A.3.5 where we give a self-contained proof. Compare the corollary to the 'unglueing' that we did in example 2.2.0. (END OF PROOF)

2.2.2 Every compact metric space is homeomorphic to a fan If we define a separable metric space to be compact whenever it is totally bounded and complete (as is standard in BISH), then it is a well-known result that every compact metric space is homeomorphic to a metric fan (a fan with a metric respecting the apartness, and endowed with the metric topology, which a fortiori is refined by the apartness topology).
Conversely, in [Waa1996] it is shown in INT that on a metric fan the apartness topology coincides with the metric topology, and that every apartness fan is metrizable. More general, in INT every star-finitary apartness spread is metrizable. We retrieve this result for BISH in section 4.0.8.

2.3 MORPHISMS ON SPREADS AND SPRAIDS

2.3.0 Refinement versus trail morphisms 1 We return briefly to our discussion of refinement morphisms versus trail morphisms. With spreads (which derive from a tree) there is no need for trail morphisms. In fact a spread $(\mathcal{V}, \mathcal{T}_\#)$ is $\preceq$-isomorphic to its trail space $(\mathcal{V}^\wr, \mathcal{T}_\#^\wr)$. Since Baire space is universal (2.2.0), we could develop a fruitful theory using only spreads and refinement morphisms (as is done in INT). Studying refinement morphisms also on spraids is therefore an enlargement of this fruitful theory. But when working with spraids, we sometimes need trail morphisms as well.

Already for practical ease we wish to work with spraids such as $\sigma_\mathbb{R}$ and refinement morphisms. This is relevant also, we believe, for computational purposes.[28] We do not straightaway know any theoretical applications, but the possibility to also use the $\preceq$-lattice properties seems to good to pass up. A relevant discipline in this respect could possibly be constructive (topological) lattice theory, or perhaps even algebraic topology.

Therefore we take some time to show that for many important spraids a trail morphism can already be directly represented by a refinement morphism. This is especially relevant from the APPLIED perspective, we believe. There is the usual drawback: it requires a bit more patience from our readers.

[28]We refer once more to the recommendations in [BauKav2009], which we follow.

2.3.1 **Unglueing of spraids** To simplify things, we show that any spraid $(\mathcal{V}, \mathcal{T}_{\#})$ can be unglued in exactly the same manner as $\sigma_{\mathbb{R}}$ in paragraph 2.2.0. The idea is to turn to the subspread of $(\mathcal{V}^{\wr}, \mathcal{T}_{\#}^{\wr})$ formed by the $\propto$-trails in $(V, \preceq)$ (instead of looking at the spread of all trails).

DEFINITION: Let $(\mathcal{V}, \mathcal{T}_{\#})$ be a spraid derived from $(V, \#, \preceq)$. The *unglueing* of $(\mathcal{V}, \mathcal{T}_{\#})$ is the spread $(\mathcal{V}^{\wr\!\wr}, \mathcal{T}_{\#}^{\wr})$ derived from the pre-natural space $(V^{\wr\!\wr}, \#^{*}, \preceq^{*})$, where $V^{\wr\!\wr} = \{a = a_0, \ldots a_{n-1} \in V^{\wr} \mid n \in \mathbb{N} \mid a$ is a $\propto$-trail and $n \geq 1 \rightarrow \lg(a_0) = 1\}$. (END OF DEFINITION)

We leave it to the reader to verify that unglueing a spraid $(\mathcal{V}, \mathcal{T}_{\#})$ amounts to adding, for each $a \in V$, a finite number of copies of a such that each $\propto$-trail from $\bigcirc$ to a is represented by one of the copies. These copies all have $\lg(a)$ as length in $(V^{\wr\!\wr}, \preceq^{*})$.
For spraids, working with $(\mathcal{V}^{\wr\!\wr}, \mathcal{T}_{\#}^{\wr})$ is more elegant than working with $(\mathcal{V}^{\wr}, \mathcal{T}_{\#}^{\wr})$, which again seems relevant for computational practice. Notice also that if we start with a spread $(\mathcal{V}, \mathcal{T}_{\#})$, then there is a trivial bijection between V and $V^{\wr\!\wr}$, showing that spreads are already unglued.

2.3.2 **Refinement versus trail morphisms 2** Now we can show that for many important spraids a trail morphism can already be directly represented by a refinement morphism. We illustrate this first with $\sigma_{\mathbb{R}}$, our preferred representation of $\mathbb{R}$.

PROPOSITION: Let f be a $\wr$-morphism from $\sigma_{\mathbb{R}}$ to $\sigma_{\mathbb{R}}$. Then there is a $\preceq$-morphism g from $\sigma_{\mathbb{R}}$ to $\sigma_{\mathbb{R}}$ such that $f(x) \equiv_{\mathbb{R}} g(x)$ for all $x \in \sigma_{\mathbb{R}}$.

PROOF: We see f as a $\preceq$-morphism from $\sigma_{\mathbb{R}}^{\wr\!\wr}$ to $\sigma_{\mathbb{R}}$. For $a \in \sigma_{\mathbb{R}}$ of the form $[\frac{4s+i}{2^{t+2}}, \frac{4s+i+2}{2^{t+2}}]$ where $1 \leq i \leq 4$ and $t \in \mathbb{N}$, put $\widehat{a} = [\frac{s}{2^t}, \frac{s+2}{2^t}]$. For all other $a \in \sigma_{\mathbb{R}}$ let $\widehat{a} = \bigcirc_{\mathbb{R}}$. Now for $a \in \sigma_{\mathbb{R}}$ there are finitely many $\propto$-trails from $\bigcirc_{\mathbb{R}}$ to a, say $b_0, \ldots, b_n$ where each b_i is in $\sigma_{\mathbb{R}}^{\wr\!\wr}$. Since the $f(b_i)$'s all touch, $\widehat{\bigcap_i f(b_i)}$ is in $\sigma_{\mathbb{R}}$. We put $g(a) \underset{D}{=} \bigcap_i \widehat{f(b_i)}$. Then g thus defined is a $\preceq$-morphism from $\sigma_{\mathbb{R}}$ to $\sigma_{\mathbb{R}}$ such that $f(x) \equiv_{\mathbb{R}} g(x)$ for all $x \in \sigma_{\mathbb{R}}$. (END OF PROOF)

COROLLARY: Let f be a $\wr$-morphism from $\mathbb{R}_{\mathrm{nat}}$ to $\mathbb{R}_{\mathrm{nat}}$. Then there is a $\preceq$-morphism g from $\mathbb{R}_{\mathrm{nat}}$ to $\mathbb{R}_{\mathrm{nat}}$ (in fact $\sigma_{\mathbb{R}}$) such that $f(x) \equiv_{\mathbb{R}} g(x)$ for all $x \in \mathbb{R}_{\mathrm{nat}}$.

PROOF: For $a \in \mathrm{R}_{\mathbb{Q}}$ let $h(a)$ be the (unique) smallest interval in $\sigma_{\mathbb{R}}$ such that $a \preceq_{\mathbb{R}} h(a)$. This determines a $\preceq$-isomorphism h from $\mathbb{R}_{\mathrm{nat}}$ to $\sigma_{\mathbb{R}}$. Again, for

$a \in \sigma_{\mathbb{R}}^{\between}$ put $\tilde{f}(a) = h(f(a))$, which yields a $\wr$-morphism $\tilde{f}$ from $\sigma_{\mathbb{R}}$ to $\sigma_{\mathbb{R}}$. By the proposition, $\tilde{f}$ can be represented by a $\preceq$-morphism g'. Now for $a \in \mathrm{R}_{\mathbb{Q}}$ simply put $g(a) = g'(h(a))$ to obtain the required $\preceq$-morphism g representing f. (END OF PROOF)

The proposition combined with paragraphs 3.4.3 and 4.0.10 illustrates that for many complete metric spaces, we can find efficient spraid representations such that we can always work with refinement morphisms.

REMARK: One relevant property here is that for a finite intersection of basic dots we can find a basic dot of ‘small enough diameter’ which contains the intersection in its interior. For our standard basic-open complete metric spreads this is obvious, but these spreads are themselves not an efficient representation. See 4.0.10. (END OF REMARK)

2.4 A DIRECT LINK WITH INTUITIONISM

2.4.0 **Natural spaces mirror Brouwer’s spreads** Our two goals in this section are to establish fundamental properties of natural spaces, and to give a classical mathematician a simple picture of important intuitionistic results. For the latter goal we establish that there is a constructive one-on-one correspondence between Brouwer’s notion of spreads, and our notion of spreads as natural spaces given by a tree (for sake of comparison let us call this notion ‘natural spread’). Brouwer’s *spread-functions* correspond precisely to natural morphisms between our natural spreads. The practical advantage lies in the retrieval of many intuitionistic results for natural spaces.

We formulate the fundamental theorem in the language of [Waa1996]. To avoid cumbersome mathematical structures, we do so on the meta-level.

META-THEOREM: Brouwer’s universal spread σ_ω corresponds precisely to natural Baire space. Other intuitionistic spreads can all be seen as an apartness spread $(\sigma, \#_\sigma)$ which can be translated *directly* to a corresponding natural spread $(\mathcal{V}_\sigma, \mathcal{T}_\#)$, which is homeomorphic in INT to $(\sigma, \#_\sigma)$ equipped with the apartness topology $\mathcal{T}_{\#_\sigma}$. Spread-functions between two spreads correspond precisely to $\preceq$-morphisms between the two corresponding natural spreads. Intuitionistic fans correspond precisely to our natural fans.

PROOF: The theorem is self-evident for anyone familiar with intuitionistic apartness topology, which was developed in [Waa1996]. For reasons of space efficiency, we do not repeat all the relevant definitions here. (END OF PROOF)

For CLASS, the theorem offers a quite direct translation of many results in intuitionistic topology to the context of natural spaces (see the remark below). In this monograph we also translate some intuitionistic results to BISH, which is usually a bit more work. At the end of this section we translate a result of Veldman, which has its place in the intuitionistic study of the Borel hierarchy (see [Vel2008] and [Vel2009]).[29] In chapter four, we translate an intuitionistic metrization theorem.

REMARK: The 'direct' translations for CLASS often co-depend on a classically valid axiom of INT called **BT** ('Brouwer's Thesis, or equivalently decidable-bar induction $\mathbf{BI_D}$) which we did not mention earlier, for simplicity. The validity of **BT** has been questioned in other branches of constructive mathematics, not in the least because Kleene showed that it fails in the branch of recursive mathematics called RUSS. In RUSS the principal axiom is **CT** (derived from 'Church's Thesis') which states that every infinite sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$ is computed by a known Turing-algorithm. Notice that this does not resemble our description of the engineer/scientist taking ever-more refinable measurements from nature.

Classical mathematicians are often unaware that compactness (more specific, the Heine-Borel property) of Cantor space fails in RUSS. This compactness does however follow from **BT**, therefore compactness can be dealt with elegantly in INT. In the past decades constructive mathematicians have put much effort in developing a variety of constructive theories which respect RUSS but still allow for a working theory of compactness. (END OF REMARK)

2.4.1 Brouwer's Thesis, compactness and induction

We discuss Brouwer's Thesis already here, although it more properly belongs in the next sections, because it influences our presentation from here on. In fact one can see **BT** as a transfinite induction scheme (countable-ordinal induction) combined with the meta-insight that we can construct open covers of Baire space only by such a transfinite induction procedure. That is, if Baire space also contains sequences about which we know only finite initial segments at any

[29] Other results of Veldman on this hierarchy might also be translatable to our classically valid setting, but we have not studied this.

given point in time. This explains why **BT** fails in RUSS, since in RUSS we have a lot of additional information about sequences (namely the algorithms computing them) and we can use this information to construct non-inductive covers of recursive Baire space. This follows from Kleene's construction of an infinite decidable subset of $\{0,1\}^*$ which contains no infinite recursive path (the Kleene Tree, see 2.5.3 and [Bau2006]). We will need some definitions to phrase **BT**.

DEFINITION: We use natural Baire space $(\mathcal{N}, \mathcal{T}_{\#_\omega})$, derived from $(\mathbb{N}^*, \#_\omega, \preceq_\omega)$. Let $A, B \subseteq \mathbb{N}^*$, then B is called a *bar* on A iff $\forall x \in [A]\, \exists n \in \mathbb{N}\, \exists b \in B\, [x_n \preceq b]$, where $[A] = \bigcup_{a \in A} [a]$. Notice that a bar on A is the same as an open cover of $[A]$ consisting of basic open sets.

Next we introduce 'genetic bars' on $\mathbb{N}^*$, using a form of countable-ordinal induction.

- $\mathbf{G}^{\blacktriangleleft}_{\bigcirc}$ The set $\{\bigcirc_\omega\}$ is a genetic bar on $\mathbb{N}^*$.
- $\mathbf{G}^{\blacktriangleleft}_{\propto}$ If for each $n \in \mathbb{N}$ we have a genetic bar B_n, then $B = \{n \star a \mid a \in B_n, n \in \mathbb{N}\}$ is also a genetic bar on $\mathbb{N}^*$.

Repeated application of the rules $\mathbf{G}^{\blacktriangleleft}_{\bigcirc}$ and $\mathbf{G}^{\blacktriangleleft}_{\propto}$ yields all genetic bars on $\mathbb{N}^*$. (END OF DEFINITION)

We believe this form of induction is constructively acceptable, and formulate the appropriate axiom:

$\boxed{\textbf{PGI}}$ The definition of genetic bars is valid. Moreover, let P be a property of bars on $\mathbb{N}^*$ such that:

- $\mathbf{G}_{\bigcirc}$ The genetic bar $\{\bigcirc_\omega\}$ has property P.
- $\mathbf{G}_{\propto}$ If for each $n \in \mathbb{N}$ we have a genetic bar B_n with property P then the genetic bar $B = \{n \star a \mid a \in B_n, n \in \mathbb{N}\}$ also has property P.

Then all genetic bars on $\mathbb{N}^*$ have property P.

DEFINITION: Let $C, D \subseteq \mathbb{N}^*$ be bars on $\mathbb{N}^*$. We say that C *descends* from D iff for each $d \in D$ there is a $c \in C$ with $d \preceq c$. (END OF DEFINITION)

REMARK: This terminology makes sense if we picture Baire space as an infinite tree, which branches *upward* from its maximal element $\bigcirc_\omega$. Now 'C descends from D' describes the picture of a bar D on $\mathbb{N}^*$, such that below each element of D there is already an element of C (therefore C covers at least what D covers). But we acknowledge that we could have called $\bigcirc_\omega$ the minimal element, reversing the $\preceq$-notation. (END OF REMARK)

We can now formulate our version of **BT**: [30]

$\boxed{\textbf{BT}}$ **PGI** holds, and every bar on $\mathbb{N}^*$ descends from a genetic bar on $\mathbb{N}^*$.

An intuitionistic plea for **BT** can be give in the following way. The universe of Baire space that we have in mind is inhabited by choice sequences arising step by step in the course of time. This means that in general only the minimum of information about an element is known. For the author, the axiom expresses that — given such a universe — the only one way to convince ourselves that a subset B of $\mathbb{N}^*$ is indeed a bar, is to show that it descends from something that we can intuitively grasp as a bar, namely a genetic bar.

In fact the more precise analysis is that there are two basic methods that can be employed in ascertaining the 'bar' status of a given subset B of $\mathbb{N}^*$. The direct method is to have B given as a genetic bar. The other method is to see that B descends from a previously ascertained bar. These two methods put together yield the method of checking whether B descends from a genetic bar.

Kleene calls this aptly 'reversing the arrows'. The genetic definition in fact mirrors what our intuition tries to do when visualising an arbitrary bar. The author has no trouble accepting this definition, and in this acceptance lies his intuitive justification of **PGI**. Brouwer's justification looks rather more complex even when explained by Veldman, but we believe it to be essentially the same as our presentation above. In [Kle&Ves1965], Kleene gives some nice examples of bars descending from genetic bars which cannot be shown to be genetic unless we prove e.g. that there are 99 consecutive 9's in the decimal expansion of π.

BT can be proved classically by contradiction as follows: suppose B is a bar on $\mathbb{N}^*$ which does not contain a genetic bar. For $a \in \mathbb{N}^*$ write B^a for $\{d \in \mathbb{N}^* \mid a \star d \in B\}$. Since B does not contain a genetic bar, $\bigcirc_\omega$ is not in B. Put $b_0 = \bigcirc_\omega$. Then for at least one $b = b_1 \in \propto(b_0)$ the bar B^b on $\mathbb{N}^*$ does not contain a genetic bar, meaning $\bigcirc_\omega \notin B^b$ which implies $b_1 \notin B$. Repetition of this argument yields a shrinking sequence $\beta = (b_n)_{n \in \mathbb{N}}$ such that β evades B. Contradiction, since B is a bar. Therefore B must contain a genetic bar, and so trivially descends from a genetic bar.

[30] This version is easily seen to be equivalent to the version given in [Waa2005].

REMARK: If one is willing to accept **BT**, then an elegant constructive theory of compactness is possible for natural spaces from the ingredients presented up to now. As stated earlier, this theory closely resembles intuitionistic results. The author believes that in physics, the (mathematics underlying the) universe which we study resembles the setting of sequences arising step by step without much further information. This is why he would especially invite physicists to take notice of intuitionistic mathematics. (END OF REMARK)

For the rest of this monograph, we will not assume **BT**. Instead, we will introduce inductive definitions which capture most of **BT**. The prize we gain is that with these inductive definitions we can build a (limited) theory of compactness also in RUSS. The price we pay is that this inductive machinery is not easy, and tends to lead to somewhat involved proofs.

Another gain is perhaps more ‘bridging’ in character. Intuitionism has never been very popular, in contrast maybe to BISH and constructive formal topology. By presenting natural spaces as spraids and then developing inductive definitions, we can hopefully illuminate intuitionistic concepts and results, while still remaining in BISH.

2.4.2 Baire morphisms together do not form a natural space In our discussion so far, we have highlighted that the class of natural spaces is large, not to say vast. In fairness let us also state that there are important spaces which cannot be represented as a natural space. In the appendix (see A.5.4) we will comment on this shortly.

A revealing example in this respect is the space of all Baire morphisms, by which we mean natural morphisms from Baire space to itself. Equivalently (by thm. 1.2.2) we can see this as the space of all continuous functions from Baire space to itself. The key for our development[31], is that every Baire morphism can be coded by an element of Baire space itself. We detail this explicitly:

DEFINITION: We fix a bijection $\lessdot\,\gtrdot$ from $\mathbb{N}^*$ to $\mathbb{N}$, with inverse $\gtrdot\,\lessdot$ such that for all $a, b \in \mathbb{N}^*$ we have $\lessdot a \gtrdot \leq \lessdot a \star b \gtrdot$. Using $\gtrdot\,\lessdot$ we can see each $\alpha \in \mathbb{N}^{\mathbb{N}}$ as a sequence of pairs of basic dots $(\gtrdot n \lessdot, \gtrdot \alpha(n) \lessdot)_{n \in \mathbb{N}}$ in $\mathbb{N}^* \times \mathbb{N}^*$. We say that α is a *coded Baire morphism* iff the so obtained set $\{(\gtrdot n \lessdot, \gtrdot \alpha(n) \lessdot) \mid n \in \mathbb{N}\}$ is a Baire morphism (a natural morphism from Baire space to itself).

[31]Which is indebted to Veldman’s expositions on intuitionism, see the bibliography.

To transfer this idea to $\mathcal{N}=\mathbb{N}^{\mathbb{N}}_{\mathrm{nat}}$, remember that for $p\in\mathcal{N}$ we write p^* to denote the unique sequence $\alpha\in\mathbb{N}^{\mathbb{N}}$ corresponding to p (see def. 2.0.2). So $p^*(m)$ then denotes $\alpha(m)$ for this $\alpha\in\mathbb{N}^{\mathbb{N}}$ corresponding to p. Thus we look on p in $\mathcal{N}$ also as a sequence of pairs of basic dots $(\rhd n \lhd, \rhd p^*(n) \lhd)_{n\in\mathbb{N}}$ in $\mathbb{N}^*\times\mathbb{N}^*$. We say that p is a *star-coded Baire morphism* iff the so obtained set $\{(\rhd n \lhd, \rhd p^*(n) \lhd) \mid n\in\mathbb{N}\}$ is a Baire morphism.[32]

We define: **Mor** $\underset{D}{=} \{p\in\mathcal{N} \mid p$ is a star-coded Baire morphism$\}$. For f in **Mor**, write $\tilde{f}$ for the f-induced Baire morphism. Now if $f\#_\omega g$ for $f, g\in$ **Mor** then the induced morphisms $\tilde{f}, \tilde{g}$ define different continuous functions from Baire space to itself. (Therefore, by slight abuse of notation, we also write: 'let $\tilde{f}$ be a morphism, determine the corresponding $f\in$ **Mor**'.) (END OF DEFINITION)

The next theorem, slightly modified, is from [Vel1981] (also read [Vel2008], [Vel2009]), as is its elegant proof (based on a Cantorian diagonal argument):

THEOREM: (Wim Veldman) Let F be a Baire morphism for which $\mathrm{Ran}(F)\subseteq$ **Mor**. Then there is an $f\in$ **Mor** such that $f\# F(p)$ for every p in $\mathcal{N}$.

COROLLARY: **Mor** cannot be represented by a natural space; the space of continuous functions from Baire space to itself cannot be represented by a natural space ('the space of continuous functions from Baire space to itself is not spreadlike').

PROOF: Let p be in $\mathcal{N}$, then $F(p)\in$ **Mor** determines a Baire morphism $\widetilde{F(p)}$. Therefore for each $p\in\mathcal{N}$, we can send p to $\widetilde{F(p)}(p)$, and this function is given by a morphism z. We now construct a Baire morphism $\tilde{f}$ such that $\tilde{f}(p)\# z(p)=\widetilde{F(p)}(p)$ for all $p\in\mathcal{N}$.
Since z is a morphism, the set $B=\{b\in\mathbb{N}^* \mid z(b)\prec\bigcirc_\omega \wedge \forall c\succ b[z(c)=\bigcirc_\omega]\}$ (the set of basic dots on which z determines the first basic dot of the image) is decidable. We take care to ensure that $\tilde{f}(b)\# z(b)$ for all $b\in B$. Then for $a\notin B$: if there is $b\in B$ with $b\prec a$, we put $\tilde{f}(a)=\bigcirc_\omega$, and otherwise there is a c and a $b\in B$ with $a=b\star c$, and we put $\tilde{f}(a)=\tilde{f}(b)\star c$.
Now for p in $\mathcal{N}$, we see that the morphism $\widetilde{F(p)}$ is apart from $\tilde{f}$, because $\widetilde{F(p)}(p)=z(p)\#\tilde{f}(p)$ by construction of $\tilde{f}$. For $\tilde{f}$, determine the corresponding $f\in$ **Mor**, then $F(p)\#_\omega f$ for all $p\in\mathcal{N}$. (END OF PROOF)

[32] For each basic dot $\rhd n \lhd$, the image under the morphism is the basic dot $\rhd p^*(n) \lhd$.

2.5 COMPACTNESS, BROUWER'S THESIS AND INDUCTIVE DEFINITIONS

2.5.0 Compactness in the absence of Brouwer's Thesis From this section on we carry out the idea mentioned in 2.4.1 (where **BT** was defined): to develop a theory of compactness using inductive definitions, in the absence of **BT**. We discuss the relationship between natural topology and some other approaches to constructive mathematics, especially regarding compactness. We do not in any way claim final wisdom, since these are complex issues.[33]

2.5.1 Pointwise, pointfree or both? We take up some extra space and time to show that natural topology suits both the pointwise and pointfree perspective. In the previous chapters the pointwise approach was our central focus. But in the absence of Brouwer's Thesis **BT**, the pointfree approach is necessary for inductive definitions if one wishes to reduplicate compactness-like results. We concentrate on 'basic dots' but at the end of the day we also have the usual separable topological spaces and the pointwise perspective at our disposal.

For clarity we repeat that the results in this paper are derived within BISH, and that these results are mostly translations of existing intuitionistic results. Intuitionistic results have been called non-effective by some authors, but we believe that the difference lies mainly in the acceptance of induction axioms versus the incorporation of the induction in all the definitions. We consider this more a question of style than a paramount distinction.

2.5.2 Introduction to constructive compactness issues To understand compactness issues we turn to Cantor space ($\{0,1\}^{\mathbb{N}}$). In CLASS and INT, Cantor space is a universal separable compact space (a result due to Brouwer). By this we mean that every separable compact space is the continuous image of $\{0,1\}^{\mathbb{N}}$. This also holds in BISH, but in BISH the definition of compactness is a metrical one, see below.

Cantor space can be pictured as a dually branching subtree of Baire space $\mathbb{N}^{\mathbb{N}}$, the infinitely branching tree. Brouwer's notation for Cantor space was σ_2, and for Baire space σ_ω. In Brouwer's terminology, every finitely branching

[33] Also there are many varieties of constructive and semi-constructive topology and the author's knowledge of and insight in these varieties is very limited.

subtree of σ_ω is called a *fan*. Brouwer's compactness axiom **FT** (the Fan Theorem) states that every cover of a fan has a finite subcover (a form of the Heine-Borel property, classically equivalent to König's lemma).

However, in recursive mathematics (RUSS) there is no immediate topological definition of 'compact'. In RUSS, Baire space is in fact homeomorphic to Cantor space, due to the axiom **CT** (see A.4.9) Therefore in our theory of natural topology, the compactness of Cantor space[34] cannot be shown unless we adopt Brouwer's **FT**, or strengthen the definition of 'cover' to 'inductive cover' (thereby excluding many interesting non-inductive covers in RUSS). **FT** can be derived from **BT**, and so also holds in CLASS. Therefore our description of natural spaces for a classical mathematician leads to results closely resembling intuitionism. **BT** fails in RUSS, from which the picture often is drawn that RUSS and INT are incompatible.

Bishop, who developed a neutral constructive stance with Bishop-style mathematics (BISH), thought that Brouwer's motivation for **BT** was mystical (notwithstanding the simplifications offered by Kleene in [Kle&Ves1965]). He tried to work around the difficulties associated with topological compactness by only defining 'metrical compactness' (meaning 'totally bounded and complete', for metric spaces). In the light of the situation in RUSS, this seems a good neutral solution and it has a large number of nice applications (see [Bis&Bri1985], [Bri&Vîţ2006]). But Bishop also wanted to obtain that continuous functions are uniformly continuous on compact spaces, and being unable to prove this by lack of a compactness axiom, he added this property to the definition of continuous function. In [Waa2005] it is shown that this is practically equivalent to the adoption of **FT** (thereby BISH somewhat loses its neutral stance, and veers towards intuitionism). In reaction to [Waa2005], in [Bri&Vîţ2006] the definition of 'continuous function' has been modified back to the usual epsilon-delta one.

In formal topology a way of dealing with compactness is to look at what we will call 'inductive covers' and 'inductive morphisms', using a form of transfinite countable-ordinal induction. It is a nice solution which in a sense incorporates **BT** already in the definitions. In this way, compactness results of formal topology also have a recursive interpretation in RUSS, which is an attractive feature shared with BISH. But as in BISH, the non-inductive non-compactness of Cantor space in RUSS then may be ignored, and formal topology in this way also seems to veer towards intuitionism.[35]

[34] The property 'every open cover has a finite subcover' (the Heine-Borel property).

[35] The author finds the literature on formal topology hard to read, and to avoid mistakes refrains from a more precise mathematical comparison.

2.5.3 **Natural Cantor and Baire space are isomorphic in RUSS** From a well-known basic result from [Kle&Ves1965] we derive the equally well-known result that in RUSS natural Cantor space is isomorphic to natural Baire space.

THEOREM: In RUSS natural Cantor space is isomorphic to natural Baire space.

PROOF: From [Kle&Ves1965] we can directly define a decidable countable subset[36] $K_{\mathrm{bar}} = \{k_n \mid n \in \mathbb{N}\} \subset \{0,1\}^*$ such that $k_n \preceq_\omega k_m$ implies $n = m$ for all $n, m \in \mathbb{N}$ and such that in RUSS $\{[k] \mid k \in K_{\mathrm{bar}}\}$ is an open cover of $(\mathcal{C}, \mathcal{T}_{\mathrm{nat}})$ which has no finite subcover. We use K_{bar} to define an isomorphism f in RUSS from $(\mathcal{N}, \mathcal{T}_{\#_\omega})$ to $(\mathcal{C}, \mathcal{T}_{\mathrm{nat}})$ as follows.

Put $f(\bigcirc_\omega) = \bigcirc_\omega$. Then let $a = a_0, \ldots, a_m \in \mathbb{N}^*$. Put $f(a) = k_{a_0} \star k_{a_1} \star \ldots \star k_{a_m}$.

The verification that f is an isomorphism is not difficult and left to the reader. (END OF PROOF)

The theorem shows that with our definitions so far we cannot hope to define a natural-topological notion of compactness in RUSS. In formal topology, this is partly resolved by using a form of transfinite induction.[37] However, pointwise problems in BISH related to these compactness issues persist (see 3.4.0). Yet we will adopt this transfinite-induction strategy as well, for elegance and for purposes of informal comparison.

2.5.4 **A model of INT as part of RUSS** Usually, INT is seen as being at odds with RUSS, because of the compactness troubles in RUSS. However, it is also possible to informally model INT as a two-player game in RUSS. In this model, one can see INT as the part of RUSS where all covers are inductive. **BT** then becomes an elegant way of saying that we restrict our recursive world to all things inductive. In this model we can prove the intuitionistic continuity principle **CP**. This model has similarities to Weihrauch's TTE (type-two effectivity), but we are no expert and refrain from making direct comparisons.

We present the model of INT as part of RUSS in section 4.2.4. This model might give an argument for physics why **FT** is more valid than 'not **FT**', *even* if $\mathbf{CT}_{\mathrm{phys}}$ is seen to hold. Here $\mathbf{CT}_{\mathrm{phys}}$ stands for the statement that nature can only produce recursive sequences. Since $\mathbf{CT}_{\mathrm{phys}}$ is as of yet undecided, RUSS might be a relevant mathematical model for physics, and these compactness issues seem worthy of physicists' attention as well.

[36] derived from what Andrej Bauer in [Bau2006] aptly calls the Kleene Tree.

[37] Which for natural spaces seems equivalent to **BT**, see prp. 3.1.0.

In BISH and in constructive formal topology, a preference for **FT** sometimes seems cloaked in definitions. This leads to an exclusion of (parts of) RUSS which is not always easy to spot. We believe it more fruitful for our foundational discussion here to give RUSS a more equal place, and then study the arising topological structures in the light of different axiom systems. This is one reason for developing the concepts 'neutrally' in the previous chapters.

2.5.5 Compactness and inductivity So now we need to find a perspective on compactness and inductivity. We adopt from formal and pointfree topology the notion of (what we call) 'inductive covers' and 'inductive morphisms'. For natural spaces, Brouwer's induction scheme in our eyes is more elegant and wieldy. Therefore we derive our inductive covers from 'genetic bars' as defined in 2.4.1, only slightly generalized to arbitrary spraids. Thus emulating formal topology in a simplified way, we can interpret compactness in RUSS as well. One may however ask oneself, after all the work has been done, whether any real benefit has been gained over intuitionism. For one answer, one could look at the hopefully better understanding of the relevant issues and the relationship between the various branches of constructive mathematics. As a second answer, we also phrase a strong support of intuitionistic mathematics in paragraph A.1.2.

CHAPTER THREE

Inductivizing our definitions

We develop genetic induction based on a simplification of bar induction. Inductive covers thus defined are equivalent to 'formal inductive covers' coupled with a formal-topology style of induction.

Every subfann (including Cantor space and the real interval $[\alpha, \beta]$) has the Heine-Borel property for inductive covers (of that fann in its mother spraid).

Inductive morphisms respect inductive covers; they are uniformously continuous on metric (sub)fanns. Continuous$^{\text{BIS}}$ functions from $\mathbb{R}$ to $\mathbb{R}$ are representable by an inductive morphism. The statement that continuous$^{\text{BIS}}$ functions from $\mathbb{R}$ to $\mathbb{R}^+$ are representable by an inductive morphism from $\mathbb{R}_{\text{nat}}$ to $\mathbb{R}^+_{\text{nat}}$ is equivalent to **FT**. Pointwise problems for BISH persist, related to the reciprocal function and compactness (the relevant example of ContraCantor space is given in the appendix).

We discuss Kleene's realizability and other ways to define inductive morphisms.

Finally we define (in)finite-product spaces, and prove a BISH version of Tychonoff's theorem.

3.0 INDUCTION IN FORMAL-TOPOLOGY STYLE

3.0.0 Bootstrap method Our development strategy has a drawback: we already gave the basic definitions, and now we want to build an inductive theory. This involves revisiting the earlier basic definitions, to 'inductivize' them. We again ask some patience from the reader.

3.0.1 Basic (open) covers and per-enumerable open covers We begin our development of inductive covers by defining a basic covering relation $\lhd$ on sets of basic dots.

DEFINITION: Let $(\mathcal{V}, \mathcal{T}_\#)$ be a natural space with corresponding $(V, \#, \preceq)$. Let $A, B \subseteq V$. We say that B is a *basic cover* of A, notation $A \lhd B$ iff for all $x \in [\![A]\!] = \{y \in \mathcal{V} \mid \exists a \in A [y \preceq a]\}$ there is a $b \in B$ with $x \preceq b$. In other words, iff $[\![A]\!] \subseteq [\![B]\!]$. Notice that $\lhd$ is transitive. By extension we say that B is a basic cover of $\mathcal{V}$ iff $\{\bigcirc\} \lhd B$. Basic covers need not correspond with open sets in the topology (but for Baire space the distinction is moot), so for $A \lhd B$ we say that B is a *basic open cover* of A if in addition $[\![B]\!]$ is open in $(\mathcal{V}, \mathcal{T}_\#)$, and then we write $A \lhd_\circ B$. (END OF DEFINITION)

Notice that this definition is not 'pointfree', it relies essentially on the points in $\mathcal{V}$. The way in which we have acquired the insight 'for all $x \in [\![A]\!]$ there is a $b \in B$ with $x \preceq b$' is left unspecified. This means that in RUSS covers derived from the Kleene Tree are also basic covers, and so compactness of Cantor space cannot be derived with regard to basic covers without extra axioms. What does hold in RUSS as well as in INT and CLASS is that every open cover of Baire space is refined by an enumerable basic open cover of Baire space, which entails a form of the Lindelöf property ('every open cover has an enumerable refinement').[38] To be able to work with this important Lindelöf property in BISH as well, we translate some definitions of [Waa1996] to our setting here.

DEFINITION: Let $(\mathcal{V}, \mathcal{T}_\#)$ as above, and let $\mathcal{U} \subset \mathcal{V}$ be open in $(\mathcal{V}, \mathcal{T}_\#)$. We say that $\mathcal{U}$ is *enumerably open* iff there is an enumerable subset U of V such that $\mathcal{U} = [\![U]\!]$. Let $\Upsilon = \{\mathcal{U}_n \mid n \in \mathbb{N}\} = \{[\![U_n]\!] \mid n \in \mathbb{N}\}$ be an enumerable collection of enumerably open subsets $\mathcal{U}_n = [\![U_n]\!]$ of $\mathcal{V}$, then we say that Υ is a *per-*

[38] See axiom **BDD** in A.4.12.

enumerable open collection. If in addition Υ is an open cover of a subset $\mathcal{W}$ of $\mathcal{V}$, then we say that Υ is a per-enumerable open cover of $\mathcal{W}$ in $(\mathcal{V}, \mathcal{T}_{\#})$. (END OF DEFINITION)

Per-enumerable covers have nice properties. Per-enumerable covers of metric spaces for instance allow a subordinate partition of unity, as well as a star-finite refinement (see [Waa1996]). These are powerful topological tools, which use the paracompact properties of metric spaces. Also, per-enumerable covers form a connection between basic dots in natural spaces and basic opens of formal topology. For example in $\mathbb{R}_{\mathrm{nat}}$, the open real interval $(0, 1)$, which is represented by a basic open in formal topology, is enumerably open since it is represented by the enumerable set of basic dots $\{[p, q] \in \mathrm{R}_{\mathbb{Q}} \mid 0 < p < q < 1\}$.

The next steps in our development concern inductive basic covers. If we specify covers inductively, we capture a form of compactness, namely the Heine-Borel property that inductive covers of a fann have a finite subcover.

3.0.2 Formal inductive covers To facilitate a connection with formal topology, we first define a 'formal' inductive covering relation $\blacktriangleleft$, using a form of transfinite countable-ordinal induction which is generally considered constructive. We later give a more concise induction scheme which is in essence due to Brouwer.

DEFINITION: Let $b, c \in V$ and $A, B, C \subseteq V$ as previously, we define:

$\mathbf{Ind}_1^{\blacktriangleleft}$ $b \preceq c$ implies $\{b\} \blacktriangleleft_V \{c\}$.[39]

$\mathbf{Ind}_2^{\blacktriangleleft}$ if for all $a \in A$ we have $\{a\} \blacktriangleleft_V B$, then $A \blacktriangleleft_V B$.

$\mathbf{Ind}_3^{\blacktriangleleft}$ if $A \blacktriangleleft_V B \subseteq C$ then $A \blacktriangleleft_V C$.

$\mathbf{Ind}_4^{\blacktriangleleft}$ if $A \blacktriangleleft_V B \blacktriangleleft_V C$ then $A \blacktriangleleft_V C$.

$\mathbf{Ind}_5^{\blacktriangleleft}$ $\{b\} \blacktriangleleft_V \{d \mid d \prec b\}$.

Repeated application of the rules $\mathbf{Ind}_1^{\blacktriangleleft}$ through $\mathbf{Ind}_5^{\blacktriangleleft}$ yields all sets $D, E \subseteq V$ for which $D \blacktriangleleft_V E$.[40] We say that B is a *formal inductive cover* of A w.r.t. $(\mathcal{V}, \mathcal{T}_{\#})$, iff $A \blacktriangleleft_V B$ (as an exercise the reader may prove by induction that $A \blacktriangleleft B$ implies

[39] One can replace this with the seemingly stronger rule: if $A \lhd B$ and B is finite, then $A \blacktriangleleft B$. We believe it equivalent, but this equivalence is a bit circuitous, depending on Baire space as universal space, so one may prefer this stronger rule.

[40] This is transfinite countable-ordinal induction

$A \lhd B$). If the context space is clear, we omit the subscript and simply write $A \blacktriangleleft B$. By extension we say that B is a formal inductive cover of $\mathcal{V}$ iff $\{\bigcirc\} \blacktriangleleft_{\mathcal{V}} B$. (END OF DEFINITION)

We state the appropriate axiom which from now on we take to hold:

$\boxed{\textbf{PFI}}$ (Principle of Formal Induction): For any natural space $(\mathcal{V}, \mathcal{T}_{\#})$, the definition of formal inductive covers is valid. Moreover, let P be a property of pairs of subsets A, B of V, such that:

Ind$_1$ $b \preceq c$ implies $P(\{b\}, \{c\})$ for $b, c \in V$.

Ind$_2$ if for all $a \in A$ we have $P(\{a\}, B)$, then $P(A, B)$.

Ind$_3$ if $P(A, B)$ and $B \subseteq C$ then $P(A, C)$.

Ind$_4$ if $P(A, B)$ and $P(B, C)$ then $P(A, C)$.

Ind$_5$ $P(\{b\}, \{d \mid d \prec b\})$ for all $b \in V$.

Then $A \blacktriangleleft B$ implies $P(A, B)$ (for $A, B \subseteq V$).

To show how things work in this formal-topology style of induction, we deduce the Heine-Borel property for formal inductive covers of Cantor space -on its own- as a corollary to the following proposition:

PROPOSITION: In $\mathcal{C}$, let $A, B \subseteq \{0, 1\}^*$ such that $A \blacktriangleleft B$. Then for all $a \in A$ there is a finite $C \subseteq B$ such that $\{a\} \blacktriangleleft C$.

PROOF: By formal induction, using as property $P(A, B)$: 'for all $a \in A$ there is a finite $C \subseteq B$ such that $\{a\} \blacktriangleleft C$'. We check that **Ind**$_1$ through **Ind**$_5$ hold for P:

Ind$_1$ trivially, by **Ind**$^{\blacktriangleleft}_1$, $b \preceq c$ implies $P(\{b\}, \{c\})$ for $b, c \in \{0, 1\}^*$.

Ind$_2$ if for all $a \in A$ we have $P(\{a\}, B)$, then for $a \in A$ we know: for all $a' \in \{a\}$ there is a finite $C' \subseteq B$ such that $\{a'\} \blacktriangleleft C'$. It trivially follows that for all $a \in A$ there is a finite $C \subseteq B$ such that $\{a\} \blacktriangleleft B$.

Ind$_3$ if $P(A, B)$ and $B \subseteq C$ then for all $a \in A$ there is a finite $D \subseteq B$ such that $\{a\} \blacktriangleleft D$. Since $B \subseteq C$ we see that $P(A, C)$.

Ind$_4$ if $P(A, B)$ and $P(B, C)$ then for arbitrarily given $a \in A$ there is a finite $E \subseteq B$ such that $\{a\} \blacktriangleleft E$. Since $P(B, C)$, we also have that for $e \in E$ there is a finite $D_e \subseteq C$ such that $\{e\} \blacktriangleleft D_e$. But then, taking $D = \bigcup_{e \in E} D_e$, we see that $D \subseteq C$ is finite and $\{a\} \blacktriangleleft E \blacktriangleleft D$, so by **Ind**$^{\blacktriangleleft}_4$ also $\{a\} \blacktriangleleft D$. Since a is arbitrary we conclude $P(A, C)$.

Ind$_5$ let $b \in \{0,1\}^*$, we wish to show $P(\{b\}, \{d \in \{0,1\}^* \mid d \prec b\})$. For this we take the finite subset $C = \{b \star 0, b \star 1\}$ of $B = \{d \mid d \prec b\}$, and we see that $\{b\} \blacktriangleleft C$, since $B \blacktriangleleft C$ by **Ind$_2^\blacktriangleleft$** and **Ind$_1^\blacktriangleleft$**, and $\{b\} \blacktriangleleft B$ by **Ind$_5^\blacktriangleleft$**, and so $\{b\} \blacktriangleleft C$ by **Ind$_4^\blacktriangleleft$**.

Therefore by **PFI**, for any $A, B \subseteq \{0,1\}^*$ such that $A \blacktriangleleft B$ we have $P(A, B)$. In other words: for all $a \in A$ there is a finite $D \subseteq C$ such that $\{a\} \blacktriangleleft D$, so we are done. (END OF PROOF)

COROLLARY:

(i) In $\mathcal{C}$, let $A \subseteq \{0,1\}^*$ be finite and $B \subseteq \{0,1\}^*$ such that $A \blacktriangleleft B$. Then there is a finite $B' \subseteq B$ such that $A \blacktriangleleft B'$.

(ii) In $\mathcal{C}$, let $B \subseteq \{0,1\}^*$ be a formal inductive cover of $\mathcal{C}$, then B has a finite subcover. (Heine-Borel property for formal inductive covers of Cantor space).

PROOF: Ad (i): In $\mathcal{C}$, by the theorem, for all $a \in A$ there is a finite $B_a \subseteq B$ such that $\{a\} \blacktriangleleft B_a$. Since A is finite, we can take $B' = \bigcup_{a \in A} B_a$.
Ad (ii): take $A = \{\bigcirc_\omega\}$ and use (i) above. (END OF PROOF)

3.1 GENETIC INDUCTION IN BROUWER'S STYLE

3.1.0 Inductive covers following Brouwer's Thesis We could continue developing the theory in this formal-topology style, but we feel that for the setting of natural spaces, Brouwer's approach is more precise and concise. Therefore we develop an alternative notion of 'inductive cover' for spraids (and prove that it amounts to the same as 'formal inductive cover'). To facilitate the connection with intuitionism, we adopt (and adapt) intuitionistic terminology. To be foundationally clear, we formalize our countable-ordinal induction scheme as an axiom (**PGI***). The advantage to this alternative approach is that definitions and proofs become shorter, and that we can more easily adopt results from intuitionism.

We generalize the Baire space definition 2.4.1 of genetic bars, genetic induction and **BT** to arbitrary spraids. (Remember from 2.2.0 that without loss of generality a natural space is given by a spraid.).

DEFINITION: Let $(\mathcal{V},\mathcal{T}_{\#})$ be a spraid with corresponding $(V,\#,\preceq)$ (so $(V,\preceq)$ is a trea). Let $B\subseteq V$. If B is a basic cover of $(\mathcal{V},\mathcal{T}_{\#})$ (see def. 3.0.2), then we say that B is a *bar* on V (in $(\mathcal{V},\mathcal{T}_{\#})$), equivalently $\{\bigcirc\}\lhd B$. We remind the reader that for any a in V, the basic subspraid $V_a=\{b\in V\,|\,b\preceq a\}=\{a\}_{\preceq}$ is formed by putting its maximal dot as $\bigcirc_a=a$.[41] We extend this notation by putting $C_{\preceq}=\bigcup_{c\in C}V_c$ for any subset $C\subseteq V$. Also kindly remember that for a in V, we write $\propto(a)$ for $\{b\in V\,|\,b\propto a\}$.

Now we inductively define genetic bars on basic subspraids V_a of $(\mathcal{V},\mathcal{T}_{\#})$ as follows:

$\mathbf{G}^{\star}_{\bigcirc}$ For $a\in V$ the set $\{\bigcirc_a\}$ is a genetic bar on V_a.

$\mathbf{G}^{\star}_{\propto}$ If for $a\in V$ and all $b\in\propto(a)$, B_b is a genetic bar on V_b, then $\bigcup_{b\in\propto(a)}B_b$ is a genetic bar on V_a.

Repeated application of the rules $\mathbf{G}^{\star}_{\bigcirc}$ and $\mathbf{G}^{\star}_{\propto}$ yields all genetic bars on basic subspraids of $(\mathcal{V},\mathcal{T}_{\#})$.[42] (END OF DEFINITION)

We state the appropriate axiom which from now on we take to hold:

$\boxed{\textbf{PGI}^*}$ (generalized Principle of Genetic Induction): For any spraid $(\mathcal{V},\mathcal{T}_{\#})$, the definition of genetic bars is valid. Moreover, let P be a property of bars on basic subspraids of $(\mathcal{V},\mathcal{T}_{\#})$ such that:

$\mathbf{G}_{\bigcirc}$ For $a\in V$, the genetic bar $\{\bigcirc_a\}$ on V_a has property P.

$\mathbf{G}_{\propto}$ If for $a\in V$ and all $b\in\propto(a)$, B_b is a genetic bar on V_b with property P, then the genetic bar $\bigcup_{b\in\propto(a)}B_b$ on V_a has property P.

Then all genetic bars on basic subspraids of $(\mathcal{V},\mathcal{T}_{\#})$ have property P.

DEFINITION: Let $(\mathcal{V},\mathcal{T}_{\#})$ as above, with $a\in V$, and let $B,C,D\subseteq V$ be bars on V_a. Then C *descends* from D iff for all $d\in D$ there is $c\in C$ with $d\preceq c$ (iff $D\subseteq C_{\preceq}$).
We say that B is an *inductive bar* on V_a (and an *inductive cover* of V_a) iff there is a genetic bar G on V_a such that B descends from G. By extension we say that B is an inductive cover of $\{a\}$, notation $\{a\}\blacktriangleleft^{\propto}_{V}B$ or simply $\{a\}\blacktriangleleft^{\propto}B$ when the context is clear. Notice that B need not be a subset of V_a.
Next, let $E,F\subseteq V$. We say that E is an inductive cover of F, notation $F\blacktriangleleft^{\propto}_{V}E$, iff for all $a\in F$ we have $\{a\}\blacktriangleleft^{\propto}_{V}E$. We also write $F\blacktriangleleft^{\propto}E$ when the context is clear.

[41] Taking $a=\bigcirc_V$ gives the entire spraid V ($(\mathcal{V},\mathcal{T}_{\#})$) itself.

[42] Again this is countable-ordinal transfinite induction. Also note that the version for Baire space is even more elegant. But using this more elegant form requires the encoding of all entities as natural numbers, which we have waived.

From a pointwise perspective (see def. 3.0.1) a per-enumerable open cover $\Upsilon = \{[U_n] \mid n \in \mathbb{N}\}$ of a subspraid $\mathcal{W} \subseteq \mathcal{V}$ derived from $(W, \#, \preceq)$ is called an *inductive open cover* of $\mathcal{W}$ iff $W \blacktriangleleft^{\propto}_{W} \bigcup_{n\in\mathbb{N}} U_n$.Then, for an arbitrary subset $\mathcal{A} \subseteq \mathcal{W}$, we also say Υ is an inductive open cover of $\mathcal{A}$. See A.5.6 for additional comments. (END OF DEFINITION)

3.1.1 Genetic and formal covers coincide on spraids We can show that for spraids there is no distinction between inductive covers (obtained through genetic bars in Brouwer's style) and formal inductive covers (defined in 3.0.2 in formal-topology style).

THEOREM: Let $(\mathcal{V}, \mathcal{T}_{\#})$ as above, and let $E, F \subseteq V$. Then $F \blacktriangleleft E$ iff $F \blacktriangleleft^{\propto} E$.

PROOF: See appendix A.3.6, we use **PFI** and **PGI***. (END OF PROOF)

REMARK: This will allow us to use $\mathbf{Ind}_1^{\blacktriangleleft}$ through $\mathbf{Ind}_5^{\blacktriangleleft}$ as properties of $\blacktriangleleft^{\propto}$ as well. The theorem hopefully also partly clarifies the relation between INT and formal topology. As explained earlier, we prefer genetic induction on spraids. (END OF REMARK)

3.1.2 Brouwer's Thesis generalized to spraids We do not adopt **BT** in our narrative, yet we cannot escape generalizing this axiom to arbitrary spraids:

$\boxed{\mathbf{BT}^*}$ **PGI*** holds, and every bar on a spraid $(\mathcal{V}, \mathcal{T}_{\#})$ descends from a genetic bar on $(\mathcal{V}, \mathcal{T}_{\#})$.

The defense of **BT*** derives straightforwardly from the defense of **BT** given in 2.4.1. **BT*** also follows from **BT**, and therefore holds in CLASS and INT.

We see **BT*** as a deep insight in the nature of how we can construct bars at all, if we have to deal with infinite sequences (of basic dots) about which at any given time we know only initial finite segments. **BT*** fails in RUSS since in RUSS we have a finite algorithm for each infinite sequence, giving us far more knowledge of such sequences than in the limited-information setting that Brouwer had in mind. We transpose Brouwer's setting to RUSS in section 4.2.4, to show what we mean.

PROPOSITION: Let $(\mathcal{V}, \mathcal{T}_{\#})$ be a spraid derived from $(V, \#, \preceq)$. Then **BT*** implies that every cover of V is inductive.

REMARK: The proof is trivial. The proposition underlines that if we accept **BT*** we can simply work with basic covers and skip the inductivizing of basic definitions. The resulting theory is common ground of CLASS and INT, and elegant. The genetic induction scheme remains essential in this theory though, so not all the work done here is superfluous if one accepts **BT***. Using **PFI** and theorem 2.2.0 ('every natural space is spreadlike') we obtain the following corollary. (END OF REMARK)

COROLLARY: Let $(\mathcal{V}, \mathcal{T}_{\#})$ be a natural space derived from $(V, \#, \preceq)$. Then **BT*** (with **PFI**) implies that every cover of V is formal-inductive.

3.1.3 Genetic bars are decidable

As a first exercise in genetic induction, we prove that a genetic bar on a spraid $(\mathcal{V}, \mathcal{T}_{\#})$ is a decidable subset of the set of basic dots V. [43]

PROPOSITION: Let $(\mathcal{V}, \mathcal{T}_{\#})$ be a spraid, with corresponding $(V, \#, \preceq)$. If B is a genetic bar on a basic subspraid V_a of $(\mathcal{V}, \mathcal{T}_{\#})$, then B is a decidable subset of V_a.

PROOF: By genetic induction, using **PGI***:

- $\mathbf{G}_{\bigcirc}$ For $a \in V$, the genetic bar $\{\bigcirc_a\}$ on V_a is a decidable subset of V_a.
- $\mathbf{G}_{\propto}$ For $a \in V$, let for all $b \in \propto(a)$, B_b be a genetic bar on V_b which is also a decidable subset of V_b. Then $\bigcup_{b \in \propto(a)} B_b$ is a genetic bar on V_a which is a decidable subset of V_a since for c in V the set $D = \{b \in \propto(a) \mid c \preceq b\} = \{b \in \propto(a) \mid c \in V_b\}$ is finite, so we can determine if $c \in \bigcup_{b \in D} B_b$ or not. And $c \in \bigcup_{b \in \propto(a)} B_b$ is equivalent to $c \in \bigcup_{b \in D} B_b$.

Therefore all genetic bars on basic subspraids V_a of $(\mathcal{V}, \mathcal{T}_{\#})$ are decidable subsets of V_a. (END OF PROOF)

REMARK: Genetic bars on spreads are decidable thin bars, where a thin bar is a bar B for which if $a \in B$ and $b \prec a$ then $b \notin B$. With **BT** one can also show the converse, that every decidable thin bar is genetic.[44] On spraids, due to the glue, genetic bars need not be thin. We will show that we can unglue genetic bars also. (END OF REMARK)

[43] This reflects our preference for genetic bars as a vehicle for countable-ordinal induction, since we feel that genetic bars have an intuitively manageable complexity.

[44] See [Waa2005], or do it yourself (nice exercise).

3.2 INDUCTIVE HEINE-BOREL FOR (SUB)FANNS

3.2.0 Inductive Heine-Borel for fanns As a second exercise in genetic induction, we first show an inductive Heine-Borel property for fanns, considered as natural space on their own. This can also be considered an inductive version of the fan theorem **FT**.

THEOREM: ($\mathbf{FT}_{\propto}$) Every genetic bar on (a basic subspraid of) a fann $(\mathcal{V}, \mathcal{T}_{\#})$ derived from $(V, \#, \preceq)$ is finite.

PROOF: Using genetic induction:

- $\mathbf{G}_{\bigcirc}$ For $a \in V$, the genetic bar $\{\bigcirc_a\}$ on V_a is finite.
- $\mathbf{G}_{\propto}$ For $a \in V$, if for all $b \in \propto(a)$, B_b is a finite genetic bar on the fann V_b, then $\bigcup_{b \in \propto(a)} B_b$ is a finite genetic bar on V_a since $\propto(a)$ is finite since $(\mathcal{V}, \mathcal{T}_{\#})$ is a fann.

Therefore all genetic bars on basic subspraids of $(\mathcal{V}, \mathcal{T}_{\#})$ are finite. (END OF PROOF)

COROLLARY: (Inductive Heine-Borel for fanns) Every inductive cover of a fann $(\mathcal{V}, \mathcal{T}_{\#})$ has a finite subcover, and (from the pointwise perspective:) every inductive open cover of $(\mathcal{V}, \mathcal{T}_{\#})$ has a finite open subcover.

PROOF: By the theorem, every inductive cover C of V descends from a finite genetic bar B (meaning: for all $b \in B$ there is a $c \in C$ with $b \preceq c$), so we find a finite subset C' of C which is already a basic cover of V. From the pointwise perspective, let $\Upsilon = \{[\![U_n]\!] \mid n \in \mathbb{N}\}$ be an inductive open cover of $\mathcal{V}$, then $C = \bigcup_{n \in \mathbb{N}} U_n$ is an inductive cover of V (see def. 3.1.0), therefore we find a finite $C' = \{c_i \mid i \leq N\} \subseteq C$ which is a basic cover of V. For each $i \leq N$ we can determine an n_i such that $c_i \in U_{n_i}$, and so $\{[\![U_{n_i}]\!] \mid i \leq N\}$ is a finite open cover of $\mathcal{V}$. (END OF PROOF)

3.2.1 Inductive Heine-Borel for subfanns (including the real interval $[\alpha, \beta]$) Mostly however, we are interested in fanns as subspraids of larger spaces. We obtain an inductive Heine-Borel property for subfanns as the corollary of a basic proposition about genetic bars on subspraids:

PROPOSITION: Let $(\mathcal{W}, \mathcal{T}_{\#})$ be a subspraid (derived from $(W, \#, \preceq)$) of a spraid $(\mathcal{V}, \mathcal{T}_{\#})$ derived from $(V, \#, \preceq)$. Let $a \in W$ and let B be a genetic bar on V_a. Then B contains a genetic bar on the basic subspraid W_a of $(\mathcal{W}, \mathcal{T}_{\#})$.

PROOF: By genetic induction:

- $\mathbf{G}_{\bigcirc}$ $B = \{\bigcirc_a\}$, then we are done.
- $\mathbf{G}_{\propto}$ Else, $B = \bigcup_{b \in \propto(a)} B_b$ where for each $b \in \propto(a)$ the genetic bar B_b on V_b is such that $b \in W$ implies that B_b contains a genetic bar C_b on W_b. This means that $C = \bigcup_{b \in \propto(a) \cap W} C_b$ is a genetic bar on W_a contained in B.

(END OF PROOF)

COROLLARY:

(i) If $E \subseteq W$ and $F \subseteq V$ and $E \blacktriangleleft^{\propto}_{V} F$ then $E \blacktriangleleft^{\propto}_{W} (F_{\preceq} \cap W)$.

(ii) (Inductive Heine-Borel for subfanns, $\mathbf{HB}_{\propto}$) If $(\mathcal{W}, \mathcal{T}_{\#})$ is a subfann (derived from $(W, \#, \preceq)$) of a spraid $(\mathcal{V}, \mathcal{T}_{\#})$, and C is an inductive cover of W in $(\mathcal{V}, \mathcal{T}_{\#})$, then C contains a finite subcover of W in $(\mathcal{V}, \mathcal{T}_{\#})$. From the pointwise perspective, if Υ is an inductive open cover of $\mathcal{W}$ in $(\mathcal{V}, \mathcal{T}_{\#})$, then Υ contains a finite open cover of $\mathcal{W}$ in $(\mathcal{V}, \mathcal{T}_{\#})$.

PROOF: Ad (i): Let $E \blacktriangleleft^{\propto}_{V} F$, then for $e \in E$ there is a genetic bar B on V_e such that F descends from B, which means that $B \subseteq F_{\preceq}$. By the proposition B contains a genetic bar C on W_e, and trivially $C \subseteq (F_{\preceq} \cap W)$. Therefore $\{e\} \blacktriangleleft^{\propto}_{W} (F_{\preceq} \cap W)$ and since e is arbitrary we see that $E \blacktriangleleft^{\propto}_{W} (F_{\preceq} \cap W)$.

Ad (ii): Determine $c = \bigcirc_W \in V$. Under conditions as stated, there is a genetic bar B on V_c such that C descends from B. By the proposition, B contains a genetic bar B' on the fann $W_c = W$ which forms $(\mathcal{W}, \mathcal{T}_{\#})$. By theorem 3.2.0 B' is finite. Since $B' \subseteq B$ and C descends from B, we find that for all $b \in B'$ there is a $c \in C$ with $b \preceq c$. So we find a finite $C' \subseteq C$ such that C' is a cover of W (in $(\mathcal{V}, \mathcal{T}_{\#})$).
From the pointwise perspective, let $\Upsilon = \{[\![U_n]\!] \mid n \in \mathbb{N}\}$ be an inductive open cover of $\mathcal{W}$, then $C = \bigcup_{n \in \mathbb{N}} U_n$ is an inductive cover of W (see def. 3.1.0), therefore we find a finite $C' = \{c_i \mid i \leq N\} \subseteq C$ which is a basic cover of W. For each $i \leq N$ we can determine an n_i such that $c_i \in U_{n_i}$, and so $\{[\![U_{n_i}]\!] \mid i \leq N\}$ is a finite open cover of $\mathcal{W}$ in $(\mathcal{V}, \mathcal{T}_{\#})$. (END OF PROOF)

REMARK: From this we conclude that for $\alpha < \beta \in \mathbb{R}$ the real interval $[\alpha, \beta] \subset \mathbb{R}_{\mathrm{nat}}$ has the inductive Heine-Borel property, also from the pointwise perspective.[45] This because it is fairly easy to indicate a finitely branching full subtrea W of $(\sigma_{\mathbb{R}}, \preceq_{\mathbb{R}})$ such that $[\alpha,\beta] = [\![W]\!]$. Therefore, any inductive (open) cover of $[\alpha,\beta]$ is an inductive (open) cover of the subfann $\mathcal{W}$ derived from W, and so contains a finite (open) cover of $[\alpha,\beta]$. (END OF REMARK)

3.3 INDUCTIVE MORPHISMS

3.3.0 Inductive morphisms: definition Having established some basic properties of inductive covers, we turn to inductive morphisms. Inductive morphisms are those morphisms which respect inductively acquired covers, inversely (looking at the pre-image). Therefore they inversely preserve inductive Heine-Borel properties. In metric spaces this implies that inductive morphisms are uniformly continuous on compact subspaces.

In order to deal with trail morphisms elegantly, we turn to the unglueing of $(\mathcal{V}, \mathcal{T}_{\#})$. But the reader can safely concentrate on refinement morphisms, as we show later on.

DEFINITION: Let f be a $\preceq$-morphism between the spraids $(\mathcal{V}, \mathcal{T}_{\#})$ and $(\mathcal{W}, \mathcal{T}_{\#_2})$, derived from $(V, \#, \preceq)$ and $(W, \#_2, \preceq_2)$. Let g be a $\wr$-morphism from $(\mathcal{V}, \mathcal{T}_{\#})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$ (so g is a $\preceq$-morphism from $(\mathcal{V}^{\taurus}, \mathcal{T}_{\#}^{\wr})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$)[46]. Recall the definition of $\mathbf{id}_*$ in the proof of theorem 1.1.4.

(i) For $c \in W$ put $\overleftarrow{f}(c) = \{b \in V \mid f(b) \preceq_2 c\}$, for $C \subseteq W$ put $\overleftarrow{f}(C) \underset{D}{=} \bigcup_{c \in C} \overleftarrow{f}(c)$.

(ii) For $c \in W$ put $\overleftarrow{g}_*(c) = \mathbf{id}_*(\overleftarrow{g}(c)) = \{b \in V \mid \exists b' \in V^{\taurus}[\mathbf{id}_*(b') = b \wedge g(b') \preceq_2 c\}$, for $C \subseteq W$ put $\overleftarrow{g}_*(C) \underset{D}{=} \bigcup_{c \in C} \overleftarrow{g}_*(c)$.

(iii) We call f resp. g an *inductive morphism* iff for any genetic bar G on W we have that $\overleftarrow{f}(G)$ resp. $\overleftarrow{g}_*(G)$ contains a genetic bar H on V.

(iv) If the context is clear, we also simply write $\overleftarrow{g}$ for $\overleftarrow{g}_*$.

(END OF DEFINITION)

[45] A similar pointfree result is proved in [Neg&Ced1996] for the formal interval $[\alpha,\beta]$. The author hasn't come across a general proof of the Heine-Borel property for 'fanlike' formal-topological subspaces, but his knowledge of formal topology is limited. A related monograph in many aspects is [M-Löf1970].

[46] Any $\preceq$-morphism from $(\mathcal{V}^{\wr}, \mathcal{T}_{\#}^{\wr})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$ is determined by its restriction to $(\mathcal{V}^{\taurus}, \mathcal{T}_{\#}^{\wr})$.

PROPOSITION: With notation as above, we have that g is an inductive trail morphism iff g is inductive as a $\preceq$-morphism from $(\mathcal{V}^{\eth}, \mathcal{T}_{\#}^{\wr})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$.

PROOF: In the appendix A.3.7 we show more, namely that genetic bars on V correspond to genetic bars on $V^{\eth}$ in a precise way. We could call this the unglueing of genetic bars on V. (END OF PROOF)

The proposition together with its proof in A.3.7 shows that we can safely restrict ourselves from now on to refinement morphisms. If at any time we need to use (inductive) trail morphisms, then we know that there is a direct correspondence between (inductive) trail morphisms on $(\mathcal{V}, \mathcal{T}_{\#})$ and (inductive) refinement morphisms on $(\mathcal{V}^{\eth}, \mathcal{T}_{\#}^{\wr})$. This explains our next:

CONVENTION: From now on, unless stated otherwise explicitly, our morphisms are refinement morphisms. (END OF CONVENTION)

Next we intend to show that if f is an inductive morphism between the spraids $(\mathcal{V}, \mathcal{T}_{\#_1})$ and $(\mathcal{W}, \mathcal{T}_{\#_2})$, then $A \blacktriangleleft^{\propto}_{W} B$ implies $\overleftarrow{f}(A) \blacktriangleleft^{\propto}_{V} \overleftarrow{f}(B)$, for subsets $A, B \subseteq W$. This is relatively straightforward, if we first tackle a few extra details about genetic bars (which are nice enough in their own right).[47]

3.3.1 Genetic bars are extendable and reducible Another basic property of genetic bars is that for $c \preceq a \in V$ we can reduce a genetic bar on V_a to a genetic bar on V_c, and vice versa if we have a genetic bar on V_c then we can expand it to a genetic bar on V_a. We need some minor technicalities for this.

LEMMA: Let $(\mathcal{V}, \mathcal{T}_{\#})$ be a spraid with corresponding $(V, \#, \preceq)$. Let $a \in V$, then for all $n \in \mathbb{N}$ the set $V_a^n = \{b \in V_a \mid \lg(b) = \lg(a) + n\}$ is a genetic bar on V_a.

PROOF: By induction on $n \in \mathbb{N}$. If $n = 0$ then $V_a^n = \{a\} = \{\bigcirc_a\}$ so we are done. Suppose the lemma holds for given n (and for all $a \in V$), then we show it holds for $n+1$ as well. For then we know that for each $c \in \propto(a)$ the set V_c^n is a genetic bar on V_c. And so by $\mathbf{G}^{\star}_{\propto}$ (see def. 3.1.0) $V_a^{n+1} = \bigcup_{c \in \propto(a)} V_c^n$ is a genetic bar on V_a. (END OF PROOF)

For $c \prec a \in V$, the basic subspraid V_c is of course covered elementarily by $\{a\}$

[47] One reason to to proceed like this is once more to keep our concepts intuitively manageable. Instead of using 'for all subsets A, B such that $A \blacktriangleleft^{\propto} B$' we only use decidable genetic bars. But it also provides for shorter proofs.

which in this respect fulfills the same maximal role as $\{\bigcirc_c\}=\{c\}$. But (for reasons of elegance) $\{a\}$ is not a genetic bar on V_c. We resolve this by introducing the 'reduction' of a genetic bar on V_a to V_c.

DEFINITION: Let $(\mathcal{V},\mathcal{T}_\#)$ be a spraid, and $c\preceq a\in V$. If B is a genetic bar on V_a, then $B^{\uparrow c}=B\cup\{\bigcirc_c\mid\exists b\in B[c\preceq b]\}$ is called the *reduction* of B to V_c. If B is a genetic bar on V_c then $B^{\downarrow a}=\{b\in V_a\mid \lg(b)=\lg(c)\wedge b\neq c\}\cup B$ is called the *expansion* of B to V_a. (END OF DEFINITION)

PROPOSITION: Let $(\mathcal{V},\mathcal{T}_\#)$ be a spraid derived from $(V,\#,\preceq)$. Let $c\preceq a\in V$.

(i) If B is a genetic bar on V_a then $B^{\uparrow c}$, the reduction of B to V_c, contains a genetic bar on V_c.

(ii) If B is a genetic bar on V_c then $B^{\downarrow a}$, the expansion of B to V_a, is a genetic bar on V_a

PROOF: Ad (i): If $a=c$ then we are done. Else, $c\prec a$. Then by genetic induction:

G$_\bigcirc$ If $B=\{\bigcirc_a\}$, then $B^{\uparrow c}=\{\bigcirc_a,\bigcirc_c\}$ and so contains the genetic bar $\bigcirc_c$ on V_c.

G$_\propto$ Else $B=\bigcup_{b\in\propto(a)}B_b$ where for all $b\in\propto(a)$, B_b is a genetic bar on V_b such that if $c\preceq b$, then $B_b^{\uparrow c}$ contains a genetic bar on V_c. There has to be at least one $b\in\propto(a)$ such that $c\preceq b$, so for such b we find that $B_b^{\uparrow c}$ contains a genetic bar on V_c. And so $B^{\uparrow c}$ also contains a genetic bar on V_c.

Ad (ii): By induction on $n=\lg(c)-\lg(a)$. If $n=0$ then $c=a$ and we are done. Suppose the lemma holds for given n (and for all $a,c\in V$) then we show it holds for $n+1$ as well. So then let $\lg(c)=\lg(a)+(n+1)$. We can locate a $b\in\propto(a)$ such that $c\preceq b$. Then by induction $B^{\downarrow b}$ is a genetic bar on V_b. By the above lemma, for all $d\in\propto(a), d\neq b$ we know that V_d^n is a genetic bar on V_d. From this we conclude that $V^{\downarrow a}=\bigcup_{d\in\propto(a),d\neq b}V_d^n\cup B^{\downarrow b}$ is a genetic bar on V_a. (END OF PROOF)

For spreads, one can define $B^{\uparrow c}$ in such a way that it is itself a genetic bar.

3.3.2 Inductive morphisms respect inductive covers (and Heine-Borel)

We need a preparatory lemma, after which we can prove the basic theorem with respect to inductive morphisms. The lemma shows that for an inductive morphism, we can transfer the inductive pre-image property of the whole space to the basic subspraids. (We could have taken that as definition too, but we believe that in practice it leads to longer proofs.)

LEMMA: Let f be an inductive morphism between the two spraids $(\mathcal{V}, \mathcal{T}_{\#_1})$ and $(\mathcal{W}, \mathcal{T}_{\#_2})$, with corresponding pre-natural spaces $(V, \#_1, \preceq_1)$ and $(W, \#_2, \preceq_2)$. Let $a \in W$, and let G be a genetic bar on W_a. Then for all $d \in V$: if $f(d) \in W_a$, then $\overleftarrow{f}(G)$ contains a genetic bar on V_d.

PROOF: The proof is surprisingly involved, we give it in A.3.8. The reader is welcome to try her/himself, it should be a good exercise. (END OF PROOF)

THEOREM: Let f be an inductive morphism between the spraids $(\mathcal{V}, \mathcal{T}_{\#_1})$ and $(\mathcal{W}, \mathcal{T}_{\#_2})$, and let $A, B \subseteq W$ where $A \blacktriangleleft^{\propto}_{W} B$. Then $\overleftarrow{f}(A) \blacktriangleleft^{\propto}_{V} \overleftarrow{f}(B)$.

PROOF: Let $a \in A$, then since $A \blacktriangleleft^{\propto}_{W} B$ there is a genetic bar G on W_a such that B descends from G. Now consider $d \in \overleftarrow{f}(a) \subseteq V$, which is equivalent to $f(d) \in W_a$. By the previous lemma $\overleftarrow{f}(G)$ contains a genetic bar on V_d. Since $\overleftarrow{f}(G) \subseteq \overleftarrow{f}(B)$ we see that $\overleftarrow{f}(B)$ contains a genetic bar on V_d, so $\{d\} \blacktriangleleft^{\propto}_{V} \overleftarrow{f}(B)$. Since a, d are arbitrary, we find that $\overleftarrow{f}(A) \blacktriangleleft^{\propto}_{V} \overleftarrow{f}(B)$. (END OF PROOF)

COROLLARY:

(i) If Z forms a subspraid $(\mathcal{Z}, \mathcal{T}_{\#_1})$ of $(\mathcal{V}, \mathcal{T}_{\#_1})$, then the restriction f_Z of f to Z is an inductive morphism from $(\mathcal{Z}, \mathcal{T}_{\#_1})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$.

(ii) If K forms a subfann $(\mathcal{K}, \mathcal{T}_{\#_1})$ of $(\mathcal{V}, \mathcal{T}_{\#_1})$, then $f(K)$ is contained in the trea E of a subfann $\mathcal{E}$ of $(\mathcal{W}, \mathcal{T}_{\#_2})$, where $f(\mathcal{K})$ is dense in $\mathcal{E}$ (so $f(\mathcal{K}) \subseteq \mathcal{E}$ but equality is not always the case).

PROOF: Ad (i): it suffices to show that $\overleftarrow{f}_Z(A) \blacktriangleleft^{\propto}_{Z} \overleftarrow{f}_Z(B)$ for $A \blacktriangleleft^{\propto}_{W} B$. We already know by the theorem that $\overleftarrow{f}(A) \blacktriangleleft^{\propto}_{V} \overleftarrow{f}(B)$. Therefore trivially $\overleftarrow{f}_Z(A) \blacktriangleleft^{\propto}_{V} \overleftarrow{f}(B)$. By corollary 3.2.1(i) we find that $\overleftarrow{f}_Z(A) \blacktriangleleft^{\propto}_{Z} ((\overleftarrow{f}(B))_{\preceq} \cap Z) = \overleftarrow{f}_Z(B)$, and we are done.

Ad (ii): by (i) above we can take (the restriction of) f to be an inductive morphism from $(\mathcal{K}, \mathcal{T}_{\#_1})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$. Determine $d = f(\bigcirc_K)$. To see that $f(K)$ is a contained in the trea E of a subfann $\mathcal{E}$ of $(\mathcal{W}, \mathcal{T}_{\#_2})$, consider for each $n \in \mathbb{N}$ the genetic bar $G_n = \{b \in W_d \mid \lg(b) = \lg(d) + n\}$ on W_d (see lemma 3.3.1). For each $n \in \mathbb{N}$ we have $\{d\} \blacktriangleleft^{\propto}_{W} G_n$, so by the theorem $\{\bigcirc_K\} \blacktriangleleft^{\propto}_{K} \overleftarrow{f}(G_n)$. Since K is a fann, there is a finite genetic bar D_n on K such that $\overleftarrow{f}(G_n)$ descends from D_n. Therefore we can finitely determine the subset $E_n = \{e \in G_n \mid \exists a \in K\, [f(a) \preceq e]\}$ of G_n, since E_n equals $\{e \in G_n \mid \exists d \in D_n\, [f(d) \preceq e]\}$. Now we can simply take $E = \bigcup_{n \in \mathbb{N}} E_n$ to fulfill the corollary. (END OF PROOF)

To develop constructive mathematics without Brouwer's Thesis **BT** (and/or the weaker fan theorem **FT**), one hopes that inductive covers and inductive morphisms enable reproducing much of classical compactness. We indicate some problems with this approach later on, but first we turn to the question which continuous functions between topological spaces are easily seen to be representable by an inductive morphism.

3.3.3 Bishop-continuous $\mathbb{R}$-to-$\mathbb{R}$-functions are inductively representable In formal topology (see [Pal2005]) a continuous$^{\text{BIS}}$ function from $\mathbb{R}$ to $\mathbb{R}$ is representable by a formal mapping from the formal reals to the formal reals, and vice versa each such mapping represents a continuous$^{\text{BIS}}$ function. We repeat this insight in our setting, but first we give the definition of 'continuous$^{\text{BIS}}$' for real-valued functions on $\mathbb{R}$.

DEFINITION: (in the pointwise setting of BISH) Let f be a function from $\mathbb{R}$ to $\mathbb{R}$, then f is *continuous*^{BIS} iff f is uniformly continuous on every BISH-compact (meaning complete and totally bounded) subspace of $\mathbb{R}$. (Equivalently, iff f is uniformly continuous on every closed interval $[-n, n]$ for $n \in \mathbb{N}$). (END OF DEFINITION)

PROPOSITION: Let f be a continuous$^{\text{BIS}}$ function from $\mathbb{R}$ to $\mathbb{R}$. Then there is an inductive morphism f^* from $\sigma_{\mathbb{R}}$ to $\sigma_{\mathbb{R}}$ such that for all $x \in \sigma_{\mathbb{R}}$ we have $f(x) \equiv f^*(x)$ (where we identify $\mathbb{R}$ and $\sigma_{\mathbb{R}}$ for convenience). Conversely, if g is an inductive morphism from $\sigma_{\mathbb{R}}$ to $\sigma_{\mathbb{R}}$, then as a function g is uniformly continuous on each compact subspace of $\mathbb{R}$.

PROOF: See appendix A.3.9. (END OF PROOF)

REMARK: If we replace the image space by $\mathbb{R}^+$, then the situation is quite different. The statement that every uniformly continuous function from $[0, 1]$ to $\mathbb{R}^+$ is representable by an inductive morphism from $[0, 1]$ to $\mathbb{R}^+$ (as a natural space) is equivalent to the fan theorem **FT**. (END OF REMARK)

For the above remark to be precise, we first need to define the positive reals $\mathbb{R}^+$ as a a natural space. We likewise define the apart-from-zero reals $\mathbb{R}^{\#0}_{\text{nat}}$.

DEFINITION: Let $\mathrm{R}^+_{\mathbb{Q}} = \{[a, b] \in \mathrm{R}_{\mathbb{Q}} \mid a > 0\} \cup \{\bigcirc_{\mathbb{R}}\}$. The space of the natural *positive real numbers* $\mathbb{R}^+_{\text{nat}}$ is the natural subspace of $\mathbb{R}_{\text{nat}}$ derived from the pre-natural space $(\mathrm{R}^+_{\mathbb{Q}}, \preceq_{\mathbb{R}}, \#_{\mathbb{R}})$. Put $\mathrm{R}^{\#0}_{\mathbb{Q}} \underset{D}{=} \{[a, b] \in \mathrm{R}_{\mathbb{Q}} \mid b < 0 \vee a > 0\} \cup \{\bigcirc_{\mathbb{R}}\}$.

The space of the natural *apart-from-zero real numbers* $\mathbb{R}^{\#0}_{\mathrm{nat}}$ is the natural subspace of $\mathbb{R}_{\mathrm{nat}}$ derived from the pre-natural space $(\mathrm{R}^{\#0}_{\mathbb{Q}}, \preceq_{\mathbb{R}}, \#_{\mathbb{R}})$.

Next, put $\sigma_{\mathbb{R}^+} \underset{D}{=} \{[a,b]\in\sigma_{\mathbb{R}} \mid a>0\}\cup\{\bigcirc_{\mathbb{R}}\}$ and $\sigma_{\mathbb{R}^{\#0}} \underset{D}{=} \{[a,b]\in\sigma_{\mathbb{R}} \mid b<0 \vee a>0\}\cup\{\bigcirc_{\mathbb{R}}\}$, and we see that $\sigma_{\mathbb{R}^+}$ and $\sigma_{\mathbb{R}^{\#0}}$ are spraids representing $\mathbb{R}^{+}_{\mathrm{nat}}$ and $\mathbb{R}^{\#0}_{\mathrm{nat}}$ respectively. (END OF DEFINITION)

LEMMA: The statement that every uniformly continuous function from $[0,1]$ to $\mathbb{R}^+$ is representable by an inductive morphism from $\sigma_{[0,1]}$ to $\sigma_{\mathbb{R}^+}$ is equivalent to the fan theorem **FT**.

PROOF: This follows from the well-known result in [Jul&Ric1984] that **FT** is equivalent to the statement that each uniformly continuous f from $[0,1]$ to $\mathbb{R}^+$ is bounded away from 0. For completeness we detail this easy consequence in the appendix A.3.9. (END OF PROOF)

This lemma foreshadows paragraph 3.4.0, where we discuss pointwise problems for BISH which arise from the absence of **BT**.

3.3.4 Inductive Baire morphisms are constructible As another positive example, we can construct inductive Baire morphisms in the following way. In steps, we build an inductive morphism f by determining first for all $\alpha\in\mathcal{N}$ a nontrivial first value $f(\alpha)(1)$ of the inductive morphism f. For this we need a genetic bar $B_{\bigcirc}$ on $\mathcal{N}$, then for each $a\succ b\in B_{\bigcirc}$ we put $f(a)=\bigcirc$, and to each $b\in B_{\bigcirc}$ we assign an $n_b\in\mathbb{N}$ and put $f(b)=n_b$. Next, for all $\alpha\in\mathcal{N}$ we determine a nontrivial refinement $f(\alpha)(2)$ of $f(\alpha)(1)$. We do so by constructing, for each $b\in B_{\bigcirc}$, a genetic bar B_b. Then we assign to each $c\in B_b$ an $n_c\in\mathbb{N}$ and put $f(c)=f(b)\star n_c$ (and for all $c\preceq d\preceq b$ we put $f(d)=f(b)$). And so on...

Although laborious, the process above really is a construction. We can show that it yields an inductive morphism and that all inductive Baire morphisms are equivalent to a morphism which is constructed in the above way. (This is another partial explanation of our preference for genetic bars. Genetic bars can in our eyes be constructed in an intuitively manageable inductive way.

3.3.5 Kleene's realizability, **BT*** and inductive morphisms The situation for other spraids is complicated by the apartness relation. But if we are willing to accept Brouwer's Thesis (**BT**) then any morphism is inductive. The reader can ponder on the question whether it is more elegant to inductivize all the definitions (see also the rest of this section, because we are still not done yet) or to accept **BT***. With **BT***, most of the inductive machinery that we developed here becomes superfluous.

A different indication of **BT**'s constructive content comes from Kleene's realizability results on intuitionistic mathematics. Kleene's formalization of intuitionistic mathematics is usually denoted FIM. After a remarkable effort, Kleene proved in [Kle1969] that if we can prove the existence of a Baire function f in FIM[48], then this function is representable by a general recursive Baire morphism $\tilde{f}$. This is sometimes called Church's Rule ($\mathbf{CR}_1$) for FIM. Notice that $\mathbf{CR}_1$ gives a true construction for the general recursive $\tilde{f}$ which is derived canonically from the existence proof of f in the formal system FIM.

The same holds for a decidable thin bar B: if we can prove its existence in FIM, then there is a recursive representative of B. In [Waa2005] it is shown that **BT** is equivalent to the combination of two axioms $\mathbf{BI_D}$ and **BDD**, where $\mathbf{BI_D}$ is Kleene's decidable-Bar Induction (see A.4.8, and ^{x}26.3 in [Kle&-Ves1965]), and **BDD** follows from $\mathbf{AC}_{10}$ (see A.4.4, A.4.12, and 27.1 in [Kle&-Ves1965]). It is also shown that the concepts of 'genetic bar' and 'decidable thin bar' coincide under assumption of **BT**.

We wish to turn this result to our advantage. Our strategy is clear: we wish to derive from a FIM existence proof of a Baire function f, a general recursive Baire morphism $\tilde{f}$ representing f such that in addition $\tilde{f}$ is inductive. We are straightaway confident that Kleene's realizability fulfills this extra aspect, by the equivalence of **BT** with the combination of the FIM-valid axioms $\mathbf{BI_D}$ and **BDD**. Therefore we think that it should be possible to prove that the general recursive Kleene-realizing morphisms are also inductive.[49]

However, we are no experts on this subject, and we can only kindly invite those who are to take this issue under consideration. As stated above, we are confident that it is possible to prove this desirable extra inductive property. So we return to our strategy. We are looking for a way to construct inductive morphisms, and if our conjecture above is true, then we have found an elegant route.

For then we can use intuitionistic theory to derive existence of a morphism f between spraids, and -if the inductivity of f is not immediately apparent already- use Kleene's realizability to construct from this 'abstract' FIM-existence proof a general recursive $\tilde{f}$ representing f which is also inductive.

[48] Equivalently we can say: 'if we can define f in FIM,...'

[49] Probably the proof should be on a meta-level, since we do not see how to formalize the genetic property within FIM. Perhaps it can be done in an easy extension of FIM, ... but the author is not knowledgeable enough in these matters.

Notice that we are still working entirely within BISH. It might by the above reasoning seem that we can avoid endorsing **BT** and the continuity principle **CP** and just incorporate the relevant conditions into our definitions. But we cannot expect this to work as easy as all that. This because in FIM, the pointwise setting plays an integral part. Information such as '$\forall x \in [0,1][f(x) > 0]$' has to be seen as having been acquired inductively, and in the absence of an endorsement of **BT**, it is not so easy to incorporate this type of information into the definitions. There is always a simple litmus test: whether the definitions work in RUSS.

3.4 POINTWISE PROBLEMS IN BISH AND FINAL INDUCTIVIZATION

3.4.0 Pointwise problems in the absence of Brouwer's Thesis We return for a moment to the discussion started in the introduction of this section. Outside of the restricted class of Bishop-locally-compact spaces[50], the property of being uniformly continuous on compact subspaces is a consequence of inductivity, but it doesn't by itself imply inductivity. In hindsight, it seems as if Bishop underestimated the necessity of an inductive machinery in order to build a smooth theory of compactness related to uniform continuity.

In [Schu2005], there seems to be a feeling that the inductive approach of formal topology solves these issues. Notwithstanding our own inductive treatment, we are not convinced that these issues are satisfactorily solved for pointwise settings such as BISH. And, our own pointfree machinery notwithstanding, we are not convinced that the pointwise setting should be abandoned in favour of the pointfree setting. Below we list some pointwise problems for BISH regarding our inductive approach, which as far as we can tell also hold for formal topology. It therefore seems to us that the conclusions in [Schu2005] are too optimistic.

META-THEOREM:
In RUSS (and by implication BISH) we have the following problems regarding pointwise use of inductive definitions:

P_1 Uniform continuity of a function f does not imply that there is an inductive morphism representing f. Counterexamples can be given even for

[50] Metric completeness is required, $\mathbb{R}^+$ and $(0,1)$ are not locally compact in BISH.

uniformly continuous functions from $[0,1]$ to $\mathbb{R}^+$. Shortly put: uniform continuity does not imply inductive representability.

P_2 Weak completeness[51] of a compact space is not preserved under inductivity. In RUSS, even for an inductive morphism from $[0,1]$ to $[0,1]$, the image of a compact subspace may be strongly incomplete.

P_3 Inductive representability is not preserved under the restriction of a function to its pointwise image space. This follows from the counterexamples for P_1, since every uniformly continuous function from $[0,1]$ to $\mathbb{R}$ is inductively representable by proposition 3.3.3. Therefore we can expect problems with the reciprocal function $x \to \frac{1}{x}$, and must continually address these problems by adapting our definitions.

Therefore in BISH, the desirable properties associated with the problems above cannot be shown to hold without further assumptions. In fact, assertion of any of these properties implies the fan theorem **FT**.

PROOF: The proof is given in the appendix A.3.11. It is derived from the construction of a compact subspace $\Im_{[0,1]}$ of $[0,1]$ such that if we write $\mathcal{C}_{[0,1]}$ for the standard embedding of Cantor space in $[0,1]$, we see: $d_{\mathbb{R}}(\Im_{[0,1]}, \mathcal{C}_{[0,1]}) = 0$ and yet in RUSS we also have $d_{\mathbb{R}}(x, \mathcal{C}_{[0,1]}) > 0$ for $\forall x \in \Im_{[0,1]}$. The 'ContraCantor space' $\Im_{[0,1]}$ is defined using the Kleene Tree. (END OF PROOF)

3.4.1 **Inductivization of the basic definitions in the absence of BT*** A final important issue for inductivity concerns our basic definitions. We started out by defining natural spaces and pre-natural spaces, using basic dots and a pre-apartness on the basic dots. If we accept **BT***, then these definitions suffice for building a classically valid theory in which compactness and inductivity are naturally incorporated, and which largely resembles INT.

However, if one wishes to build an inductive theory for natural spaces without accepting **BT***, then one has to 'inductivize' the basic definitions. (Without **BT*** pointwise problems persist though, by thm. 3.4.0.).

REMARK: To do this thoroughly it seems attractive to abandon pointwise notions, since they generally require a pointfree translation to use inductive information. This starts already with the definition of topology itself. The 'arbitrary union' requirement (**Top₃**) is pointwise in our constructive setting,

[51] The property for a located subset A of a metric space (X,d) that for all $x \in X$: if $x \# a$ for all $a \in A$, then $d(x,A) = \inf(\{d(x,a) \mid a \in A\}) > 0$.

see 1.0.5(iii). It is possible to remedy this with a pointfree notion 'toipology', and to thus develop a completely pointfree version of natural topology (say 'natural toipology'). But we believe one should remember that the concept of 'point' is actually the same as the concept of a countable sequence and the concept of countable infinity. These concepts are already heavily involved in the very definition of V. Therefore the foundational gain to the author seems smaller than one might think at first glance, and there is a price to pay in terms of readability. For this reason, we will continue here with the basic pointwise notions. (END OF REMARK)

For inductivization of the basic definitions, it suffices to exact that covers which are required to exist by the basic definitions are inductive. These covers originate from the definition of points and $\#$-open sets.

From the definition of points, for a spraid $(\mathcal{V}, \mathcal{T}_\#)$ derived from $(V, \#, \preceq)$ and basic dots $a \# b$ in V we know that $\forall x \in \mathcal{V} \exists n \in \mathbb{N} [x_n \# a \vee x_n \# b]$. This means that the subset $C = \{c \in V \mid c \# a \vee c \# b\}$ is a bar on $(\mathcal{V}, \mathcal{T}_\#)$. For a really smooth working of inductivity, we should know that C is an inductive bar, and adapt our definition accordingly.

Other covers which arise directly from our basic definitions are the ones associated with $\#$-open subsets $\mathcal{U} \subseteq \mathcal{V}$. Recall that $\mathcal{U}$ is $\#$-open iff for each $x \in \mathcal{U}$ and $y \in \mathcal{V}$ we can determine (non-exclusively) $x \# y$ and/or there is an $m \in \mathbb{N}$ with $[y_m] \subseteq \mathcal{U}$. But for given $x \in \mathcal{U}$ this is equivalent to saying that the set $B = \{b \in V \mid b \# x_{\lg(b)} \vee [b] \subseteq \mathcal{U}\}$ is a bar on V. For a really smooth working of inductivity, we should know that this bar is inductive, and we should add this to our definition of $\#$-open accordingly.

Our solution is to add the word 'inductive(ly)' to the original definitions:

DEFINITION: Let $(\mathcal{V}, \mathcal{T}_\#)$ be a spraid derived from $(V, \#, \preceq)$. An $\#$-open subset $\mathcal{U} \subseteq \mathcal{V}$ is called *inductively open* in $(\mathcal{V}, \mathcal{T}_\#)$ iff for any $x \in \mathcal{U}$ the set $B^x_{\mathcal{U}} = \{b \in V \mid b \# x_{\lg(b)} \vee [b] \subseteq \mathcal{U}\}$ is an inductive bar on V. We then write: $\mathcal{U}$ is $\propto$-open. The collection of $\propto$-open subsets of $\mathcal{V}$ is called the *inductive apartness topology* on $(\mathcal{V}, \mathcal{T}_\#)$, notation $\mathcal{T}^\propto_\#$.

We call $(\mathcal{V}, \mathcal{T}_\#)$ an *inductive* spraid (we write: 'a $\propto$-spraid') iff:

(i) For all a, b in V with $a \# b$, the set $C = \{c \in V \mid c \# a \vee c \# b\}$ is an inductive bar on V.

(ii) The inductive apartness topology coincides with the apartness topology, that is $\mathcal{T}_\# = \mathcal{T}^\propto_\#$.

Finally, some abbreviating notation will be also useful. For subsets A, B of V we write $A \# B$ iff $a \# b$ for all $a \in A, b \in B$. We write $A \approx B$ iff $a \approx b$ for some $a \in A, b \in B$. We shortly write $a \# B$, $a \approx B$ for $\{a\} \# B$, $\{a\} \approx B$ respectively. We define: ${}^{n}V \underset{D}{=} \{a \in V \mid \lg(a) = n\}$, for $n \in \mathbb{N}$. (END OF DEFINITION)

REMARK: It follows from **BT** that every spraid is inductive. We believe that any spraid which is FIM-definable will be inductive, by our remarks in 3.3.5. One easily sees for instance that Baire space and $\sigma_{\mathbb{R}}$ (the spraid representing $\mathbb{R}$) are inductive. We can prove that a metric spraid is inductive when its trea is given by shrinking metric balls as basic dots, where each dot of $\lg(n)$ is a metric ball of diameter less than 2^{-n} and apartness of dots implies a positive distance between the dots. This shows that complete metric spaces (by a standard completion procedure) are homeomorphic to an inductive spread, see paragraph 3.4.3. In the absence of **BT** it seems practical to demand inductiveness by definition. (END OF REMARK)

For completeness we still need to show that $\mathcal{T}_{\#}^{\propto}$ is indeed a topology. We add to this a simple extension of the first property of inductive spraids (we need this extended property later on in our metrization theorem):

PROPOSITION:

(i) For a spraid $(\mathcal{V}, \mathcal{T}_{\#})$ with corresponding $(V, \#, \preceq)$, the collection $\mathcal{T}_{\#}^{\propto}$ is a topology which is refined by $\mathcal{T}_{\#}$.

(ii) Let $(\mathcal{V}, \mathcal{T}_{\#})$ be an inductive spraid derived from $(V, \#, \preceq)$. Then for finite subsets $A \# B$ of V, the subset $C = \{c \in V \mid c \# A \vee c \# B\}$ is an inductive bar on $(\mathcal{V}, \mathcal{T}_{\#})$.

PROOF: The proof is a bit involved, we give it in the appendix see A.3.10. (END OF PROOF)

3.4.2 Inductivization of other definitions

In the absence of **BT***, for completeness one should also inductivize some other definitions given in earlier sections. The most important definition in this respect is the definition of 'fanlike' since in RUSS (where **BT*** fails) Baire space is fanlike under the non-inductivized definition, by proposition 2.5.3 . The isomorphism that we constructed in the proof is an example of a non-inductive morphism.
To define '$\propto$-fanlike' for spraids we could use the inductive morphisms already defined, but we wish to generalize this definition to natural spaces, and to inductivize the definition of 'spreadlike' also. In general, natural spaces

need not be given by an apartness on a trea, so we must first expand our definition of 'inductive morphism' to include such natural spaces, and for this we use the formal inductive covering relation $\blacktriangleleft$ (see def. 3.0.2) in the obvious way.

DEFINITION: Let f be a morphism from a natural space $(\mathcal{V}, \mathcal{T}_{\#_1})$ to a natural space $(\mathcal{W}, \mathcal{T}_{\#_2})$. We say that f is *inductive* iff for all $A, B \subseteq W$ where $A \blacktriangleleft_W B$ we have that $\overleftarrow{f}(A) \blacktriangleleft_V \overleftarrow{f}(B)$ (this agrees on spraids with the earlier definition by theorem 3.3.2 and proposition 3.1.0). We say that $(\mathcal{V}, \mathcal{T}_{\#})$ is $\propto$*-fanlike* resp. $\propto$*-spreadlike* iff there is an inductive isomorphism from $(\mathcal{V}, \mathcal{T}_{\#})$ to a $\propto$-fan resp. a $\propto$-spread (with an inductive inverse). (END OF DEFINITION)

We can now show: Baire space is not $\propto$-fanlike. In fact any spraid which is $\propto$-fanlike contains a subfann on which the identity is an isomorphism with the whole space, see corollary 3.3.2(ii) and its proof.

REMARK: In our BISH framework therefore, Heine-Borel compactness is best characterized as '$\propto$-fanlike'. We will use this to phrase a natural-topology version of Tychonoff's theorem in the next section. (END OF REMARK)

3.4.3 Complete metric spaces have an inductive representation

We aim to show the viability of the concept 'inductive spraid', by proving that a complete metric space has a representation as a $\propto$-spraid. (In fact it suffices to look closely at the proof of theorem 1.2.3, and adapt the natural space constructed there).

THEOREM: Every complete metric space (X, d) is homeomorphic to a $\propto$-spraid.

PROOF: We give the proof in the appendix A.3.12. The idea is not difficult: in the proof of theorem 1.2.3 we constructed, for a complete metric space (X, d), a natural space $(\mathcal{V}, \mathcal{T}_{\#})$ homeomorphic to (X, d). If we look more carefully, we see that its trail space $(\mathcal{V}^{\wr}, \mathcal{T}_{\#}^{\wr})$ contains a (homeomorphic) $\propto$-spread. (END OF PROOF)

3.5 (IN)FINITE PRODUCTS AND TYCHONOFF'S THEOREM

3.5.0 Products, lazy convergence and isolated points For a (BISH) natural-topological version of Tychonoff's theorem, we must define (in)finite-product spaces. The idea is straightforward, but there are technical issues related to our 'lazy convergence' of points, and isolated points, see also A.5.3.

Isolated points are points which as a set are open in the topology, for instance in a one-point natural space. Generally, consider a natural space $(\mathcal{V}, \mathcal{T}_{\#})$ derived from $(V, \#, \preceq)$, where $a \in V$ is 'isolated': for all $b \# c \in V$ one has $a \# b \vee a \# c$. Then both a and the infinite sequence $\underline{a} = a, a, a, \ldots$ have the $\#$-characteristics of a point, except (and this is crucial) that their image under a morphism generally does not share those point-characteristics.

DEFINITION: Let $(\mathcal{V}, \mathcal{T}_{\#})$ be a natural space derived from $(V, \#, \preceq)$. Put $\mathring{V} =_D \{a \in V \mid \forall b, c \in V[b \# c \rightarrow (a \# b \vee a \# c)]\}$, then elements of $\mathring{V}$ are *isolated* basic dots. We call $(\mathcal{V}, \mathcal{T}_{\#})$ a *decidable-isolation* space iff $\mathring{V}$ is a decidable subset of V, and a *perfect* space iff $\mathring{V} = \emptyset$. (END OF DEFINITION)

An obvious try for a definition of the product of two natural spaces $(\mathcal{V}, \mathcal{T}_{\#_1})$ and $(\mathcal{W}, \mathcal{T}_{\#_2})$ (derived from $(V, \#_1, \preceq_1)$, $(W, \#_2, \preceq_2)$) is to take $V \times W$ as the set of basic dots, and to define $(c, d) \preceq (a, b)$ iff $c \preceq_1 a \wedge d \preceq_2 b$ and $(c, d) \# (a, b)$ iff $c \#_1 a \vee d \#_2 b$. This works fine for perfect spaces $(\mathcal{V}, \mathcal{T}_{\#_1})$ and $(\mathcal{W}, \mathcal{T}_{\#_2})$. But for isolated $a \in V$ and $q \in \mathcal{W}$ the sequence $(a, q_0), (a, q_1), (a, q_2), \ldots = (\underline{a}, q)$ becomes a point in $\mathcal{V} \times \mathcal{W}$. This is unwanted, since we need the coordinate projections $\pi_0 : (x, y) \rightarrow x$ and $\pi_1 : (x, y) \rightarrow y$ to be morphisms.

3.5.1 Definition of (in)finite-product spaces We look to retain the elegance of the simple approach when possible, and yet ensure at all times that the coordinate projections are morphisms. For this we concentrate first on the most important class of spreads/spraids, since we wish the product to be a spread/spraid as well.

We could restrict ourselves to spraids w.l.o.g., but other representations interest us also. The simple approach works for perfect spaces. They form a subclass of the decidable-isolation spaces where a slightly adjusted approach works in all but trivial cases. Finally we give a general definition which works for all spaces, but is somewhat less elegant.

DEFINITION: Let $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$ be natural spaces derived from the corresponding pre-natural $((V_n,\, \#_n,\, \preceq_n))_{n\in\mathbb{N}}$ with maximal dots $(\bigcirc_n)_{n\in\mathbb{N}}$. We form $V_\Pi{}^{(n)} = \Pi_{i\leq n}V_i = \{(a_0,\ldots,a_n) \mid \forall i \leq n[a_i \in V_i]\}$ for $n\in\mathbb{N}$, and also $V_\Pi = \bigcup_{n\in\mathbb{N}} V_\Pi{}^{(n)}$.

For $n \leq m \in \mathbb{N}, a \in V_\Pi{}^{(n)}, b \in V_\Pi{}^{(m)}$ put $a \#_\Pi b$ iff there is $i \leq n$ with $a_i \#_i b_i$. Also:

$b \preceq_\Pi a$ iff $\forall i \leq n[b_i \preceq_i a_i]$
$b \prec_{\overset{\circ}{\Pi}} a$ iff $b \prec_\Pi a \wedge \forall i \leq n[a_i \in \mathring{V}_i \to b_i \prec_i a_i]$; $b \prec_{\overset{\circ\circ}{\Pi}} a$ iff $b \prec_{\overset{\circ}{\Pi}} a \wedge m > n$
$b \prec_{\overset{\bullet}{\Pi}} a$ iff $\forall i \leq n[b_i \prec_i a_i]$; $b \prec_{\overset{\bullet\bullet}{\Pi}} a$ iff $b \prec_{\overset{\bullet}{\Pi}} a \wedge m > n$

Then $\preceq_{\overset{\circ}{\Pi}}$ is decidable when $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$ are all decidable-isolation spaces. If $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$ are perfect spaces then $\preceq_{\overset{\circ}{\Pi}}$ equals $\preceq_\Pi$.

For $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$ as above the *simple finite product* $\Pi_{i\leq n}(\mathcal{V}_i, \mathcal{T}_{\#_i})$ is the natural space derived from $(V_\Pi{}^{(n)},\, \#_\Pi,\, \preceq_\Pi)$ with maximal dot $\bigcirc_\Pi^{(n)} = \bigcirc_0, \ldots, \bigcirc_n$. We also write $\mathcal{V}_0 \times \mathcal{V}_1 \ldots \times \mathcal{V}_n$ for $\Pi_{i\leq n}(\mathcal{V}_i, \mathcal{T}_{\#_i})$.
The *simple infinite product* $\Pi_{n\in\mathbb{N}}(\mathcal{V}_n, \mathcal{T}_{\#_n})$ is the natural space derived from $(V_\Pi,\, \#_\Pi,\, \preceq_\Pi)$ with maximal dot $\bigcirc_\Pi \underset{D}{=} \bigcirc_0$. We often omit the word 'simple'. For perfect spaces, the simple (in)finite products suffice.

When $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$ are spraids, then $(V_\Pi{}^{(n)},\, \preceq_\Pi)$ and $(V_\Pi,\, \preceq_\Pi)$ are treas. Yet $\Pi_{i\leq n}(\mathcal{V}_i, \mathcal{T}_{\#_i})$ and $\Pi_{n\in\mathbb{N}}(\mathcal{V}_n, \mathcal{T}_{\#_n})$ usually fail to be a spraid (also see A.5.5) since for a spraid, infinite $\prec$-trails have to define a point. So we put $V_{\Pi,\sigma}{}^{(n)} \underset{D}{=} \{a \in V_\Pi{}^{(n)} \mid \forall i,j \leq n[\lg(a_i) = \lg(a_j)]\}$, $V_{\Pi,\sigma} \underset{D}{=} \bigcup_{n\in\mathbb{N}}\{a \in V_{\Pi,\sigma}{}^{(n)} \mid \forall i \leq n[\lg(a_i) = n]\}$.

The *finite-product spraid* $\Pi^\sigma_{i\leq n}(\mathcal{V}_i, \mathcal{T}_{\#_i})$ is the spraid (or spread) derived from $(V_{\Pi,\sigma}{}^{(n)},\, \#_\Pi,\, \preceq_\Pi)$ with maximal dot $\bigcirc_\Pi^{(n)}$. We frequently write $\mathcal{V}_0 \overset{\sigma}{\times} \mathcal{V}_1 \ldots \overset{\sigma}{\times} \mathcal{V}_n$ for $\Pi^\sigma_{i\leq n}(\mathcal{V}_i, \mathcal{T}_{\#_i})$, and for $V_{\Pi,\sigma}{}^{(n)}$ we also write $V_0 \overset{\sigma}{\times} V_1 \ldots \overset{\sigma}{\times} V_n$.
The *infinite-product spraid* $\Pi^\sigma_{n\in\mathbb{N}}(\mathcal{V}_n, \mathcal{T}_{\#_n})$ is the spraid (or spread) derived from $(V_{\Pi,\sigma},\, \#_\Pi,\, \preceq_\Pi)$ with maximal dot $\bigcirc_\Pi = \bigcirc_0$.

When $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$ are decidable-isolation spaces, then the *finite* $\circ$*-product* $\mathring{\Pi}_{i\leq n}(\mathcal{V}_i, \mathcal{T}_{\#_i})$ is the natural space derived from $(V_\Pi{}^{(n)},\, \#_\Pi,\, \preceq_{\overset{\circ}{\Pi}})$ with maximal dot $\bigcirc_\Pi^{(n)}$. We also write $\mathcal{V}_0 \mathring{\times} \mathcal{V}_1 \ldots \mathring{\times} \mathcal{V}_n$ for $\Pi_{i\leq n}(\mathcal{V}_i, \mathcal{T}_{\#_i})$.
The *infinite* $\circ$*-product* $\mathring{\Pi}_{n\in\mathbb{N}}(\mathcal{V}_n, \mathcal{T}_{\#_n})$ is derived from $(V_\Pi,\, \#_\Pi,\, \preceq_{\overset{\circ\circ}{\Pi}})$ with maximal dot $\bigcirc_\Pi$. We can replace $\preceq_{\overset{\circ\circ}{\Pi}}$ with $\preceq_{\overset{\circ}{\Pi}}$ when $\forall n \in \mathbb{N} \exists m > n[\bigcirc_m \notin \mathring{V}_m]$.

For $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$ as above the *strict finite product* $\overset{\bullet}{\Pi}_{i\leq n}(\mathcal{V}_i, \mathcal{T}_{\#_i})$ is the natural space derived from $(V_\Pi{}^{(n)},\, \#_\Pi,\, \preceq_{\overset{\bullet}{\Pi}})$ with maximal dot $\bigcirc_\Pi^{(n)}$. We also frequently write $\mathcal{V}_0 \dot{\times} \mathcal{V}_1 \ldots \dot{\times} \mathcal{V}_n$ for $\Pi_{i\leq n}(\mathcal{V}_i, \mathcal{T}_{\#_i})$.
The *strict infinite product* $\overset{\bullet}{\Pi}_{n\in\mathbb{N}}(\mathcal{V}_n, \mathcal{T}_{\#_n})$ is derived from $(V_\Pi,\, \#_\Pi,\, \preceq_{\overset{\bullet\bullet}{\Pi}})$ with maximal dot $\bigcirc_\Pi$. We can replace $\preceq_{\overset{\bullet\bullet}{\Pi}}$ with $\preceq_{\overset{\bullet}{\Pi}}$ when $\forall n \in \mathbb{N} \exists m > n[\bigcirc_m \notin \mathring{V}_m]$. We often omit the word 'strict'. (END OF DEFINITION)

The reader may verify that these definitions are valid. Some key properties are stated in the next paragraphs, one of which contains a constructive form of Tychonoff's theorem.

3.5.2 Projections and the Tychonoff topology (Spaces and notation continued from the previous paragraph:) we wish to throw some light on the relation between the product-apartness topology and the Tychonoff (product) topology. Classically, the Tychonoff topology is the 'natural' product topology, it being the coarsest topology rendering the coordinate projections continuous. For natural spaces however, these projections moreover need to be morphisms. For (in)finite products of (weak) basic neighborhood spaces (by far the most important class) the two topologies coincide, which is not surprising by theorem 1.2.2. In BISH, for other spaces this remains elusive to us, although the product-apartness topology refines the Tychonoff topology. In CLASS and INT we can show that also for the (in)finite product of star-finitary spaces (see def. 4.0.10) the two topologies coincide.

Therefore, calling an (in)finite product 'faithful' if the apartness topology and the Tychonoff topology coincide, there will certainly be no easy examples of unfaithful products. Faithfulness is an important property for products, which we now define along with 'weak basic neighborhood space' and the relevant coordinate projections π_i, using the spaces and (in)finite products from the previous paragraph.

DEFINITION: (notation from 3.5.1) Let $a = a_0, \ldots, a_s \in V_\Pi$, for $i \leq s$ put $\pi_i(a) \underset{D}{=} a_i$ and for $i > s$ put $\pi_i(a) \underset{D}{=} \bigcirc_i$. For $i \leq m$ and points $x \in \Pi_{j \leq m} \mathcal{V}_j$ and $y \in \Pi_{n \in \mathbb{N}} \mathcal{V}_n$, put $x_{[i]} \underset{D}{=} \pi_i(x) = \pi_i(x_0), \pi_i(x_1), \ldots$ and $y_{[i]} \underset{D}{=} \pi_i(y) = \pi_i(y_0), \pi_i(y_1), \ldots$.

Let $(\mathcal{W}, \mathcal{T}_\#)$ be any of the products defined in 3.5.1, then $(\mathcal{W}, \mathcal{T}_\#)$ is *faithful* iff the product-apartness topology $\mathcal{T}_\#$ and the Tychonoff topology coincide.

A natural space $(\mathcal{V}, \mathcal{T}_\#)$ is a *weak basic neighborhood space* iff for every $x \in \mathcal{V}$ there is an $x' \equiv x$ such that x'_n is a basic neighborhood of x for every $n \in \mathbb{N}$ (so every basic neighborhood space is a weak basic neighborhood space by prp. 1.2.2). (END OF DEFINITION)

PROPOSITION: (about def. 3.5.1)

(i) For $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n \in \mathbb{N}}$ and the defined simple (in)finite products, the appropriate projections π_i are morphisms iff all the $\mathcal{V}_j$'s involved in the product are perfect spaces.

(ii) For the other defined (in)finite-product spaces, the appropriate projections π_i are always morphisms.

(iii) For (weak) basic-neighborhood spaces $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$, the (in)finite products are faithful (weak) basic neighborhood spaces.

(iv) In CLASS and INT, also the (in)finite products of star-finitary spaces are faithful.

(v) For other spaces, the natural topology of the (in)finite products refines the Tychonoff topology.[52]

PROOF: We leave (i) and (ii) as exercise for the reader, and prove (iv) in A.3.13. For (iii) first let $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$ be basic-open spaces. Let $n\in\mathbb{N}$, and consider $a = a_0, \ldots, a_n \in V_\Pi{}^{(n)}$. We hold that $[a]$ is open in the relevant defined products (see def. 3.5.1), since the set $[a]$ equals $\bigcap_{i\leq n} \pi_i{}^{-1}([a_i])$ where the π_i's are morphisms by (ii) (and so continuous), and the $[a_i]$'s are open. This also shows that the basic open set $[a]$ is open in the Tychonoff topology, therefore the natural topology and the Tychonoff topology coincide.

So for basic-open $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$ the relevant (in)finite products are again basic-open. For basic neighborhood spaces $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$ we can now use the coordinate-wise isomorphisms with a basic-open space to see that the relevant (in)finite products are again basic neighborhood spaces.

For weak basic neighborhood spaces $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$, a very similar argument using $\mathbf{AC}_{01}$ can be given. Let x be in an (in)finite product, then for each relevant i there is $y_{(i)} \equiv_i x_{[i]}$ with $(y_{(i)})_n$ a basic neighborhood of $x_{[i]}$ for all $n\in\mathbb{N}$. Using $\mathbf{AC}_{01}$ we can determine a $y \equiv_\Pi x$ in the same product such that $(y_{[i]})_n$ is a basic neighborhood of $x_{[i]}$ for all $n\in\mathbb{N}$. This implies that y_n is a basic neighborhood of x for all $n\in\mathbb{N}$.
If our x is in some open $\mathcal{U}$, then we find $[y_n] \subseteq \mathcal{U}$ for some $n\in\mathbb{N}$. However, $[\![y_n]\!]$ equals $\bigcap_{i\leq M} \pi_i{}^{-1}([\![\pi_i(y_n)]\!])$ for some $M\in\mathbb{N}$, and so is a neighborhood of x in the Tychonoff topology. This shows that the product is faithful.

Finally (v): for other spaces, the natural topology of the relevant (in)finite products renders the projections as morphisms and therefore continuous, so it trivially refines the Tychonoff topology. (END OF PROOF)

3.5.3 **Spraid products and Tychonoff's theorem** Our prime purpose with this section was to define (in)finite products for spraids. Having done so, we

[52] It remains elusive to us whether unfaithful products exist in CLASS, RUSS, or INT.

phrase some (expected) nice properties of these products as a theorem. There is one important property which we can so far only prove in BISH if the product is faithful, namely that if $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$ are $\propto$-spraids, then a faithful (in)finite product is also a $\propto$-spraid.

Specified to $\propto$-fanns, this yields a natural-topology version of Tychonoff's theorem (since we characterized Heine-Borel compactness as '$\propto$-fanlike' in 3.4.2). In keeping with the rest of the chapter, to show in BISH that the faithful product of inductive spraids is again inductive requires quite some work. We start with the following definition and lemma:

DEFINITION: Let $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$ be spraids derived from the corresponding pre-natural $((V_n, \#_n, \preceq_n))_{n\in\mathbb{N}}$. Given subsets $A_i \subseteq V_i$ for $i \leq n \in \mathbb{N}$, we put

$$\Pi^{\sigma}_{i\leq n} A_i = A_0 \overset{\sigma}{\times} \ldots \overset{\sigma}{\times} A_n \underset{D}{=} \{a \in V^{(n)}_{\Pi,\sigma} \mid \forall i \leq n\, [\pi_i(a) \in (A_i)_{\preceq}] \wedge \exists j \leq n\, [\pi_j(a_j) \in A_j]\}$$
$$\overline{\Pi^{\sigma}_{i\leq n} A_i} \underset{D}{=} \{a \in V_{\Pi,\sigma} \mid \forall i \in \mathbb{N}\, [\pi_i(a) \in (A_i)_{\preceq}]\}$$

(this aligns with the definition of $V \overset{\sigma}{\times} W$ in def. 3.5.1). (END OF DEFINITION)

LEMMA: (Notations as above) Let $a \in V_0, b \in V_1$ and let G, H be genetic bars on $(V_0)_a, (V_1)_b$ respectively. Then $G \overset{\sigma}{\times} H$ is an inductive bar on $(V_0)_a \overset{\sigma}{\times} (V_1)_b$.

PROOF: By double genetic induction, see A.3.14. (END OF PROOF)

COROLLARY: For $i \leq n$ let B_i be an inductive bar on V_i. Then $\Pi^{\sigma}_{i\leq n} B_i$ is an inductive bar on $V^{(n)}_{\Pi,\sigma}$ and $\overline{\Pi^{\sigma}_{i\leq n} B_i}$ is an inductive bar on $V_{\Pi,\sigma}$.

THEOREM: For (star-finite) spreads (spraids, fanns) $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$ the products $\Pi^{\sigma}_{i\leq n}(\mathcal{V}_i, \mathcal{T}_{\#_i})$ and $\Pi^{\sigma}_{n\in\mathbb{N}}(\mathcal{V}_n, \mathcal{T}_{\#_n})$ are in turn (star-finite) spreads (spraids, fanns). For $\propto$-spraids the products $\Pi^{\sigma}_{i\leq n}(\mathcal{V}_i, \mathcal{T}_{\#_i})$ and $\Pi^{\sigma}_{n\in\mathbb{N}}(\mathcal{V}_n, \mathcal{T}_{\#_n})$, if faithful, are in turn inductive.

PROOF: We prove that for $\propto$-spraids $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$ the products $\Pi^{\sigma}_{i\leq m}(\mathcal{V}_i, \mathcal{T}_{\#_i})$ and $\Pi^{\sigma}_{n\in\mathbb{N}}(\mathcal{V}_n, \mathcal{T}_{\#_n})$, if faithful, are in turn $\propto$-spraids. The other (combinations of) properties are left as an exercise for the reader.
So let $\Pi^{\sigma}_{i\leq m}(\mathcal{V}_i, \mathcal{T}_{\#_i})$ be a faithful finite product of the $\propto$-spraids $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$. To show that $\Pi^{\sigma}_{i\leq m}(\mathcal{V}_i, \mathcal{T}_{\#_i})$ is inductive, first let $a \#_{\Pi} b$ in $V^{(m)}_{\Pi,\sigma}$. We must show that the bar $B = \{c \in V^{(m)}_{\Pi,\sigma} \mid c \#_{\Pi} a \vee c \#_{\Pi} b\}$ is inductive. There is $i \leq m$ with $a_i \#_i b_i$, so $C_i = \{c \in V_i \mid c \#_i a_i \vee c \#_i b_i\}$ is an inductive bar on V_i. For $j \neq i$ let $C_j = \{\bigcirc_j\}$, then by the above lemma and corollary $C = C_0 \overset{\sigma}{\times} \ldots \overset{\sigma}{\times} C_n$ is an inductive bar on $V^{(m)}_{\Pi,\sigma}$. It is easy to see that $C \subseteq B$ so B is inductive as well.

Second, let $\mathcal{U}$ be open in $\Pi^{\sigma}_{i\leq m}(\mathcal{V}_i,\mathcal{T}_{\#_i})$. We must show that $\mathcal{U}$ is $\propto$-open. For this let $x\in\mathcal{U}$. To see that the bar $B^x_{\mathcal{U}}=\{b\in V^{(m)}_{\Pi,\sigma} \mid b\,\#_{\Pi}\,x_{\lg(b)} \vee [\![b]\!]\subseteq\mathcal{U}\}$ is an inductive bar on $V^{(m)}_{\Pi,\sigma}$, determine $\propto$-open $\mathcal{U}_i\ni x_{[i]}$ in each $\mathcal{V}_i$ such that $\bigcap_i \pi_i^{-1}(\mathcal{U}_i)\subseteq\mathcal{U}$ (remember $\Pi^{\sigma}_{i\leq m}(\mathcal{V}_i,\mathcal{T}_{\#_i})$ is faithful). Then for $i\leq m$ the bars $B^x_{\mathcal{U}_i}=\{b\in V_i \mid b\,\#_i\,(x_{[i]})_{\lg(b)} \vee [\![b]\!]\subseteq\mathcal{U}_i\}$ are inductive and monotone.

For $i\neq j\leq m$ let $D_{i,j}=\{\bigcirc_j\}$ and $D_{i,i}=B^x_{\mathcal{U}_i}$, then by the above lemma and corollary $D_i=D_{i,0}\overset{\sigma}{\times}\ldots\overset{\sigma}{\times}D_{i,m}$ is a monotone inductive bar on $V^{(m)}_{\Pi,\sigma}$. By lemma A.3.10, the bar $D=\bigcap_i D_i$ is a monotone inductive bar on $V^{(m)}_{\Pi,\sigma}$. Clearly $D\subseteq B^x_{\mathcal{U}}$, so $B^x_{\mathcal{U}}$ is inductive as well and we are done for $\Pi^{\sigma}_{i\leq m}(\mathcal{V}_i,\mathcal{T}_{\#_i})$.

Finally, let $\Pi^{\sigma}_{n\in\mathbb{N}}(\mathcal{V}_n,\mathcal{T}_{\#_n})$ be faithful. We can copy the arguments above for $\Pi^{\sigma}_{i\leq m}(\mathcal{V}_i,\mathcal{T}_{\#_i})$, replacing C,D with $\overline{C},\overline{D}$, to see that $\Pi^{\sigma}_{n\in\mathbb{N}}(\mathcal{V}_n,\mathcal{T}_{\#_n})$ is inductive. (END OF PROOF)

REMARK: Our BISH countable version of Tychonoff's theorem is that for $\propto$-fanns $((\mathcal{V}_n,\mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$ the products $\Pi^{\sigma}_{i\leq n}(\mathcal{V}_i,\mathcal{T}_{\#_i})$ and $\Pi^{\sigma}_{n\in\mathbb{N}}(\mathcal{V}_n,\mathcal{T}_{\#_n})$, if faithful, are in turn $\propto$-fanns. In CLASS and INT the inductive properties follow from **BT**, and the countable Tychonoff's theorem is easy. (END OF REMARK)

CHAPTER FOUR

Metrizability, and natural topology in physics

Metric spaces are perhaps the most important topological spaces, and the question of metrizability has been fundamental for the development of classical topology. In constructive topology, the concept of 'located in' is usually tied to a metric. We discuss the alternative '(topologically) strongly halflocated in', which is transitive. We give the basic definitions, and show first that there are interesting non-metrizable natural spaces. We can define Silva spaces also in the context of natural topology, and infinite-dimensional Silva spaces are non-metrizable.

We translate the intuitionistic metrizability of star-finitary apartness spreads to BISH, using the inductive definitions of the previous chapter. The resulting metrization theorem for star-finitary natural spaces closely resembles classical metrization results for strongly paracompact spaces. Star-finite metric developments are also of interest for efficient computing with complete metric spaces.

In the last section, we again discuss which mathematics are suited for physics. We also present an informal two-player model of INT in RUSS, called 'Limited Information for Earthlings' (LIfE).

4.0 METRIC SPACES AND (NON-)METRIZABILITY

4.0.0 Metric spaces and (non-)topological notions The best-studied topological spaces in constructive mathematics are metric spaces, since a metric gives a lot of constructive traction in defining compactness, continuity and open covers. Still, different metrics can give rise to the same topology.[53] Conversely many notions defined for a metric space are not 'topological' – that is they are not necessarily preserved under homeomorphisms. Metrical completeness is the standard example of a non-topological notion, which leads us to the definition of 'topologically complete' (namely 'homeomorphic to a complete metric space').[54]

A more important example of a non-topological constructive notion is that of a subset A being 'located' in a metric space (X, d), meaning that the distance to A can be calculated for any $x \in X$. Although the notion 'located in' is extensively used, it has substantial drawbacks especially in the context of topology. We discuss some alternatives given in [Waa1996] which resolve these drawbacks. A form of locatedness is important also for natural topology. We briefly discuss this here, to expand on in the appendix.

4.0.1 Various concepts of locatedness What are these drawbacks of the concept 'located in'? First of all, the notion is not transitive, which is unpractical when working with extensions and subspaces of (X, d). Second, even for a closed located $A \subset X$, the notion gives little handhold for $x \in X$ to find $a \in A$ such that $x \# a$ implies $x \# A$, which is an important prerequisite for many constructions involving A. Thirdly, as mentioned, the notion is non-topological and this means we cannot use it easily in the context of topology.

In [Waa1996] several alternative notions are given in BISH, of which 'strongly halflocated in' seems more fruitful than 'located in'. The notion is transitive, unlike 'located in'. To see that it gives results, note its use in the BISH-proof of the Dugundji extension theorem in [Waa1996]. Another result is that every complete metric space can be isometrically embedded in a normed linear extension such that it becomes strongly halflocated in this extension – and where we know of no general proof that it is located.

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[53] In [Waa1996] these metrics are then called 'equivalent' and 'strongly equivalent' if they give rise to the same Cauchy-sequences.

[54] E.g. 'locally compact' in BISH is a non-topological notion relying on metrical completeness. We believe this to be unwieldy for topology.

'Topologically strongly halflocated' is seen in INT to be equivalent on complete metric spaces to a topological locatedness property called 'strongly sublocated in'. This notion can also be defined for the apartness topology of natural spaces, and seems relevant. We repeat the definitions for these properties in the appendix A.3.15, in order not to lose reading pace.

4.0.2 Natural metric spaces Let us start by defining natural metric spaces, as natural spaces with a metric (respecting the apartness). The induced metric topology need not coincide with the apartness topology however. For (a suitable natural representation of) a complete metric space we can show that the apartness topology coincides with the metric topology (see thm. 1.2.3, which is analogous to $(\mathbb{R}_{\mathrm{nat}}, \mathcal{T}_{\#_{\mathbb{R}}})$ being homeomorphic to $(\mathbb{R}, d_{\mathbb{R}})$).
Weakly metrizable natural spaces are those spaces on which we can construct a metric respecting the apartness, and they are metrizable if the metric topology coincides with the apartness topology. Complete metric spaces are thus metrizable. We will see that there are weakly metrizable spaces which are non-metrizable.

DEFINITION: For a natural space $(\mathcal{W}, \mathcal{T}_{\#})$ a morphism d from $\mathcal{W} \times \mathcal{W}$ to $\mathbb{R}_{\mathrm{nat}}$ is called a *metric* on $(\mathcal{W}, \mathcal{T}_{\#})$ iff for each $\alpha, \beta, \gamma \in \mathcal{W}$ we have

(i) $d(\alpha,\beta) \equiv d(\beta,\alpha) \geq 0$

(ii) $d(\alpha,\beta) > 0$ iff $\alpha \# \beta$

(iii) $d(\alpha,\gamma) \leq d(\alpha,\beta) + d(\beta,\gamma)$

If $\mathcal{T}$ is a topology on $\mathcal{W}$, then we say that d *metrizes* $\mathcal{T}$ iff the metric topology determined by d is the same as $\mathcal{T}$. $(\mathcal{W}, \mathcal{T}_{\#})$ is called *weakly metrizable* if there is a metric d on $(\mathcal{W}, \mathcal{T}_{\#})$, and *metrizable* if this metric metrizes $\mathcal{T}_{\#}$.
(END OF DEFINITION)

In this section we show that a vast class of natural spaces is metrizable. This class is not limited to (natural representations of) locally compact spaces, in fact the 'star-finitary' property that we use resembles (strong) paracompactness very closely. Star-finite open covers are a useful tool in general topology, and accessible to constructive topology in the context of metric spaces (see [Waa1996]).

Before turning to metrizable spaces, let us consider the question whether we can find interesting and important non-metrizable natural spaces.

4.0.3 **Are there interesting non-metrizable spaces?** Apartness topology (and possible metrizability) is extensively studied in [Waa1996], in the context of INT. Still, only simple examples of non-metrizable natural spaces are given. They seem to us of little value other than for demonstration purposes.

In this monograph we do not delve into non-metrizability either, but we show that in infinite-dimensional topology the construction of direct limits leads us to interesting non-metrizable natural spaces. These spaces have already been studied in a TTE setting by others, see e.g. [Kun&Sch2005] for a nice article discussing also practical computational aspects.[55] The examples given in [Kun&Sch2005] are: the space of polynomials with real coefficients, the space of real analytic functions on $[0,1]$, and the space of distributions with compact support. It should be possible to translate these spaces to the setting of natural spaces, since the TTE treatment resembles our development, but there might be some work involved in constructivizing the classical results which are used.

Our simple example will consist of taking the direct limit of all the Euclidean spaces $(\mathbb{R}^n)_{n\in\mathbb{N}}$, which is equivalent to the space of 'eventually vanishing real sequences', for which limit we show that the resulting classical limit-topology is in fact the apartness topology, and that this topology is non-metrizable. The non-metrizability is well-known, but the equivalence to the apartness topology is a nice illustration, we hope, that the concepts in this paper are also of interest for classical mathematicians.

The space of eventually vanishing real sequences is one way of representing the space of polynomials with real coefficients. Therefore our example also illustrates how to translate the spaces in [Kun&Sch2005] to our setting.

4.0.4 **Direct limits of natural spaces and Silva spaces** We define the direct limit of an increasing sequence of natural spaces along classical lines, where the apartness topology provides a very easy way to do so.

DEFINITION: Let $(\mathcal{V},\mathcal{T}_{\#})$ be an natural space derived from $(V,\#,\preceq)$. Suppose there is a sequence of subsets $(V_n)_{n\in\mathbb{N}}$ of $V=\bigcup_{n\in\mathbb{N}}V_n$, where for each $n\in\mathbb{N}$ the pre-natural space $(V_n,\#,\preceq)$ gives rise to a natural subspace $(\mathcal{V}_n,\mathcal{T}_{\#})$ of $(\mathcal{V},\mathcal{T}_{\#})$, and where in addition $V_n\subseteq V_{n+1}$.
Then $(\mathcal{V},\mathcal{T}_{\#})$ is called the *direct limit* of $((\mathcal{V}_n,\mathcal{T}_{\#}))_{n\in\mathbb{N}}$. (END OF DEFINITION)

[55] As stated earlier TTE is not an area in which the author is knowledgeable, it would seem that results from CLASS are taken for granted, and then translated to a computability setting.

This very simple definition suffices for our purposes here, but it also is, we believe, a faithful mirror of the traditional definition for e.g. Silva spaces arising as direct limits of Banach spaces, since on a separable complete metric space the metric topology coincides with the apartness topology (see theorem 1.2.3).

4.0.5 The direct limit of the Euclidean spaces is non-metrizable Firstly, let $\overrightarrow{\mathbb{R}}^{\omega} = \{x \in \mathbb{R}^{\mathbb{N}} \mid \exists n \in \mathbb{N} \forall m \in \mathbb{N}, m \geq n [x_{[m]} \equiv 0]\}$ be the subset of $\mathbb{R}^{\mathbb{N}}$ of eventually vanishing real sequences.[56] As a subset of $\mathbb{R}^{\mathbb{N}}$, it is at least weakly metrizable, but a different topology arises when we consider $\overrightarrow{\mathbb{R}}^{\omega}$ as the direct limit of the Euclidean spaces $(\mathbb{R}^n)_{n\in\mathbb{N}}$. This direct limit is a very simple example of an infinite-dimensional Silva space (meaning the direct limit of an increasing sequence of Banach spaces, with an extra compactness condition on the closed inclusion images of the respective unit spheres). We show that this direct limit arises naturally in our context, when building $\overrightarrow{\mathbb{R}}^{\omega}$ as the point set of a natural space. Notice that $\overrightarrow{\mathbb{R}}^{\omega}$ is not a subspace of $\mathbb{R}^{\mathbb{N}}$ under our definition of 'natural subspace'.

So to build $\overrightarrow{\mathbb{R}}^{\omega}$ as a natural space, we cannot simply use the infinite-product definitions used in defining $\mathbb{R}^{\mathbb{N}}$ (see definition 3.5.1), although we can use the same collection of basic dots. We should distinguish between $\overrightarrow{\mathbb{R}}^{\omega}$ as a point-subset of $\mathbb{R}^{\mathbb{N}}$, and $\overrightarrow{\mathbb{R}}^{\omega}$ as the point set of the natural space $(\overrightarrow{\mathbb{R}}^{\omega}, \overrightarrow{\mathcal{T}}^{\omega}_{\#})$, but to avoid tedious notation we will not do so.

DEFINITION: Let $\mathsf{R}^*_{\mathbb{Q}} = (\mathsf{R}_{\mathbb{Q}})_{\pi} = \bigcup_{n\in\mathbb{N}} \mathsf{R}^n_{\mathbb{Q}}$ be the set of finite sequences of closed rational intervals. Denote the empty sequence as $\bigcirc_{\mathbb{R}^{\mathbb{N}}}$. We define $\mathbb{R}^{\mathbb{N}}_{\mathrm{nat}}$ as the infinite product $\Pi_{n\in\mathbb{N}}\mathbb{R}_{\mathrm{nat}}$, see definition 3.5.1, but we write $\#_{\mathbb{R}^{\mathbb{N}}}$ for the product apartness and $\preceq_{\mathbb{R}^{\mathbb{N}}}$ for the product refinement (and often $\mathbb{R}^{\mathbb{N}}$ for $\mathbb{R}^{\mathbb{N}}_{\mathrm{nat}}$)

To define $\overrightarrow{\mathbb{R}}^{\omega}$ as the point set of a natural space, we need to sharpen the product apartness $\#_{\mathbb{R}^{\mathbb{N}}}$ and the refinement $\preceq_{\mathbb{R}^{\mathbb{N}}}$. To understand this, note that for $n \in \mathbb{N}$ we now consider the basic dots in $\mathsf{R}^{n+1}_{\mathbb{Q}}$ as belonging to points $x \in \mathbb{R}^{\mathbb{N}}$ with $x_{[m]} \equiv 0$ for all $m > n+1$.
So let $a = a_0, \ldots, a_n \in \mathsf{R}^{n+1}_{\mathbb{Q}}$ and $b = b_0, \ldots b_m \in \mathsf{R}^{m+1}_{\mathbb{Q}}$ with $m \geq n$, then we put $a \#_{\overrightarrow{\mathbb{R}}} b$ iff: there is $i \leq n$ with $a_i \#_{\mathbb{R}} b_i$ and/or there is $n < i \leq m$ with $0 \notin b_i$ (which is decidable since b_i is a rational interval for all $i \leq m$). Similarly we put $b \preceq_{\overrightarrow{\mathbb{R}}} a$ iff: $b_i \preceq_{\mathbb{R}} a_i$ for all $i \leq n$ and $0 \in b_i$ for all $n < i \leq m$.

[56] Remember that we write $x_{[m]}$ for the real number which is the m-th coordinate of x as an infinite sequence of real numbers, and x_m for the m-th basic dot of x as a sequence of basic dots. Also, we drop the subscript 'nat' in $\mathbb{R}_{\mathrm{nat}}$ for notational simplicity.

The *natural space of eventually vanishing real sequences* is $(\overrightarrow{\mathbb{R}}^{\omega}, \overrightarrow{\mathcal{T}}^{\omega}_{\#})$, derived from the pre-natural space $(\mathrm{R}^{*}_{\mathbb{Q}}, \#_{\overrightarrow{\mathbb{R}}}, \preceq_{\overrightarrow{\mathbb{R}}})$. We also for each $n \in \mathbb{N}^{+}$ define: $\mathbb{R}^{n}_{*} = \{x \in \overrightarrow{\mathbb{R}}^{\omega} \mid \forall m \in \mathbb{N}\, [x_m \in \bigcup_{1 \leq i \leq n} \mathrm{R}^{i}_{\mathbb{Q}}]\}$. We see: if $x \in \mathbb{R}^{n}_{*}$, then $x_{[m]} \equiv 0$ for all $m > n$. Notice that there is only a trivial difference between $\mathbb{R}^{n}_{*}$ and $\mathbb{R}^{n}$ defined as finite product space in 3.5.1. (END OF DEFINITION)

PROPOSITION: $(\overrightarrow{\mathbb{R}}^{\omega}, \overrightarrow{\mathcal{T}}^{\omega}_{\#})$ is the non-metrizable direct limit of $(\mathbb{R}^{n}_{*}, \overrightarrow{\mathcal{T}}^{\omega}_{\#})_{n \in \mathbb{N}}$, where $(\mathbb{R}^{n}_{*}, \overrightarrow{\mathcal{T}}^{\omega}_{\#})$ is $\preceq$-isomorphic to the Euclidean space $(\mathbb{R}^{n}, \mathcal{T}_{\#_{\mathbb{R}}})$ for $n \in \mathbb{N}$. There is a continuous injective surjection (which does not have a continuous inverse) from $(\overrightarrow{\mathbb{R}}^{\omega}, \overrightarrow{\mathcal{T}}^{\omega}_{\#})$ to $(\overrightarrow{\mathbb{R}}^{\omega}, \mathcal{T}_{\#_{\mathbb{R}^{\mathbb{N}}}})$ as subspace of $(\mathbb{R}^{\mathbb{N}}, \mathcal{T}_{\#_{\mathbb{R}^{\mathbb{N}}}})$.

PROOF: We prove this in the appendix A.3.16, but the reader should have no problem with it as an exercise. (END OF PROOF)

REMARK:

(i) More generally one can prove in this fashion that infinite-dimensional Silva spaces are non-metrizable natural spaces. We do not go into this, but it is an easy generalization of the above constructions for $(\mathbb{R}^{n})_{n \in \mathbb{N}}$ and $\mathbb{R}^{\mathbb{N}}$.

(ii) Perhaps interesting: $\mathbb{R}_{\mathrm{nat}}$ 'equals' the direct limit of $([-n, n]^{*}_{\mathrm{rat}}, \mathcal{T}_{\#_{\mathbb{R}}})_{n \in \mathbb{N}^{+}}$, where $[-n, n]^{*}_{\mathrm{rat}} = \{[a, b] \in \mathbb{Q} \times \mathbb{Q} \mid -n \leq a < b \leq n\} \cup \{\bigcirc_{\mathbb{R}}\}$.

(END OF REMARK)

For non-metrizable natural spaces resembling examples in [Ury1925a] we refer the reader to [Waa1996]. It should be possible to transfer the separation properties T_1 through T_4 given there to our setting of natural spaces.

4.0.6 **Metrizability results for paracompact-like spaces** Metrizability of topological spaces has played a very important part in the development of classical general topology. In constructive mathematics, the focus until recent years has been primarily on metric spaces. In [Fre1937], an intuitionistic metrization result for compact spaces was obtained. This result was extended in [Tro1966] to locally compact spaces. In [Waa1996], using a different technique, these results were reobtained for apartness spreads, and metrizability was extended to 'star-finitary' spaces. The latter resembles classical metrization results for strongly paracompact spaces.

For now, we turn to the metrizability results in [Waa1996]. For completeness, let us repeat the classical definitions of 'paracompact' and 'star-finite cover'.

In CLASS, a topological space $(X, \mathcal{T})$ is called *paracompact* iff every open cover $\mathcal{U}$ of the space has a locally finite refinement, meaning a refining open cover $\mathcal{W}$ such that each $x \in X$ is contained in only finitely many open sets in $\mathcal{W}$. An open cover $\mathcal{U}$ of $(X, \mathcal{T})$ is called *star-finite* iff for each $U \in \mathcal{U}$ there are only finitely many elements $W \in \mathcal{U}$ such that $U \cap W \neq \emptyset$. So we see that a star-finite open cover is always locally finite, but the converse is not true. Therefore a space is called 'strongly paracompact' iff every open cover of the space has a star-finite refinement.

There are many classical results on metrizability in regard to paracompactness and strong paracompactness, also specifically for separable Lindelöf spaces (which resembles our setting). We profess not to have a clear assessment of the relation between our result and these classical results, although we expect the below constructive metrization theorem to follow in CLASS from an existing classical metrization theorem. The classical theorem most resembling our constructive treatment seems the one stating that a T_0 space is metrizable iff it has a development $(\mathcal{W}_n)_{n \in \mathbb{N}}$ where $\mathcal{W}_{n+1}$ is a star-finite refinement of $\mathcal{W}_n$ for each $n \in \mathbb{N}$.

4.0.7 Star-finitary spaces From the definition of spraids, it might seem at first glance that for a given spraid $(\mathcal{V}, \mathcal{T}_\#)$ derived from the pre-natural space $(V, \#, \preceq)$ and $a \in V$, there are only finitely many $b \in V$ with $\lg(b) = \lg(a)$ and $V_b \cap V_a \neq \emptyset$ (one could call this a $\lg(a)$ star-finite basic intersection property, but we will not use this). However, this does not follow from the definition if one looks carefully, and counterexamples are easy to construct.

Even if a $\lg(n)$ star-finite basic intersection property holds for each $n \in \mathbb{N}$, then because of the complicating apartness relation, it tells us little about the natural topology of the space $(\mathcal{V}, \mathcal{T}_\#)$. Therefore we will define our star-finite property in terms of the apartness relation $\#$ (or rather its complementing touch-relation $\approx$) on basic dots. For metrizability results we then look at 'star-finitary' natural spaces, which are $\propto$-isomorphic to a star-finite $\propto$-spraid $(\mathcal{V}, \mathcal{T}_\#)$ (where the inductivity of $(\mathcal{V}, \mathcal{T}_\#)$ ensures nice behaviour of the apartness relation).

DEFINITION: Let $(\mathcal{V}, \mathcal{T}_\#)$ be a spraid derived from $(V, \#, \preceq)$, where $\approx\, \subset V \times V$ is the complement of $\#$. We say that $\approx$ as well as $(\mathcal{V}, \mathcal{T}_\#)$ and $(V, \#, \preceq)$ are *star-finite* iff for each $a \in V$ the subset $\{b \in V \mid \lg(b) = \lg(a) \wedge b \approx a\}$ is finite. A natural space $(\mathcal{W}, \mathcal{T}_\#)$ is called *star-finitary* iff $(\mathcal{W}, \mathcal{T}_\#)$ is $\propto$-isomorphic to a star-finite $\propto$-spraid. (END OF DEFINITION)

The class of star-finitary spaces is large since countable products of star-finitary spaces are again star-finitary. Baire space is a star-finite $\propto$-spraid, and $\mathbb{R}^{\mathbb{N}}$ is star-finitary (a star-finite $\propto$-spraid representing $\mathbb{R}^{\mathbb{N}}$ can be obtained by forming the countable product of copies of the star-finite $\propto$-spraid $\sigma_{\mathbb{R}}$).

4.0.8 Star-finitary spaces are metrizable In this paragraph we translate an intuitionistic result regarding metrizability of apartness spreads. This translation (especially its proof) takes some time, since the original result makes multiple use of the intuitionistic axioms $\mathbf{AC}_{10}$ and **FT**. Our translation below (and especially its proof) is therefore also a good example how to circumvent the use of these axioms if one is so inclined, and simultaneously shows that -as should be expected- intuitionistic theorems carry 'effective' content. [57]

THEOREM: Every star-finitary natural space is metrizable.

PROOF: It suffices to prove the theorem for a star-finite $\propto$-spraid (remember that '$\propto$-spraid' is shorthand for 'inductive spraid'). For this we translate the proof of the corresponding intuitionistic theorem in [Waa1996] for star-finite apartness spreads. This translation is in principle unproblematic, since the use of **FT** in that proof can now be replaced by the use of the inductive Heine-Borel property for subfanns ($\mathbf{HB}_{\propto}$), and the use of $\mathbf{AC}_{10}$ becomes superfluous by our definition of inductive spraid.
Still, since the original result is not trivial, our translation also involves quite some work. We give this translation in the appendix, see A.3.18. We also sketch the main idea of the proof below, when discussing the corollary. (END OF PROOF)

COROLLARY: Every $\propto$-fann is metrizable ('every compact space is metrizable'), and every space with a one-point $\propto$-fanlike extension is metrizable ('every locally compact space is metrizable').

PROOF: For didactical reasons we will first prove the corollary directly in A.3.17, before proving all of the theorem above. Let us sketch the strategy for a $\propto$-fan $(\mathcal{W}, \mathcal{T}_{\#})$ derived from $(W, \#, \preceq)$ already here. To each $a \# b \in {}^nW$ (for certain $n \in \mathbb{N}$) we can construct a canonical morphism $f_{a,b}$ from $(\mathcal{W}, \mathcal{T}_{\#})$ to $[0,1]_{\text{ter}}$ such that $f_{a,b} \equiv_{\mathbb{R}} 0$ on W_a and $f_{a,b} \equiv_{\mathbb{R}} 1$ on W_b. Then let

[57] In [Pal2009] the usage of intuitionistic axioms is called non-effective, which indicates that their constructive content might not be immediately apparent, a reason to elucidate this constructive content here.

$h : \mathbb{N} \to W \times W$ be an enumeration of all apart pairs $a \# b \in {}^nW$ (for all $n \in \mathbb{N}$), and put $d(x, y) \underset{D}{=} \sum_{n\in\mathbb{N}} 2^{-n} \cdot |f_{h(n)}(x) - f_{h(n)}(y)|$ to obtain the desired metric d.

For $a \# b \in {}^nW$ (for certain $n \in \mathbb{N}$), in order to construct a canonical morphism $f_{a,b}$ from $(\mathcal{W}, \mathcal{T}_\#)$ to $[0,1]_{\text{ter}}$ such that $f_{a,b} \equiv_{\mathbb{R}} 0$ on W_a and $f_{a,b} \equiv_{\mathbb{R}} 1$ on W_b, we split W first in W_0, W_1 and W_2 where $a \approx W_0$, $a \# W_1 \# b$ and $W_2 \approx b$ and moreover $W_0 \# W_2$.

For this we use the splitting lemma A.3.17 (**B**), which tells us that finite apart $A \# B \subset W$ always lead (for big enough $N \in \mathbb{N}$) to a partition of ${}^NW$ in three sets C, D, E where $A \approx C$, $A \# D \# B$, $E \approx B$ and moreover $C \# E$.

Then we iterate this procedure for each W_i (where $i \in \{0, 1, 2\}$), splitting e.g. W_0 in W_{00}, W_{01}, W_{02} where $a \approx W_{00}$, $a \# W_{01} \# W_1$ and $W_{02} \approx W_1$ and moreover $W_{00} \# W_{02}$.
In this way, we actually construct a function from W to $\{0, 1, 2\}^*$, such that this function represents a morphism from $(\mathcal{W}, \mathcal{T}_\#)$ to $\sigma_{3,\text{ter}}$ (see example 1.3.2), where the latter is isomorphic to $[0,1]_{\text{ter}}$.

To generalize this strategy for $\propto$-fans to a star-finite $\propto$-spread involves a lot of extra work, but the basic idea remains the same. For the complete proofs, we refer the reader to A.3.17 and A.3.18. (END OF PROOF)

REMARK:

(i) The theorem seems to cover much of what is possible, metrization-wise. One can prove in INT (and likely CLASS, RUSS) that complete metric spaces are weakly star-finitary (which in CLASS is equivalent to being star-finitary). We discuss this in the next paragraph.

(ii) The example in A.2.1 of a possible spherical completion of $\mathbb{C}_p$ might pose an interesting application for the metrization theorem above, but we confess to not having studied this issue in any detail.

(END OF REMARK)

4.0.9 Complete metric spaces are weakly star-finitary in INT It is shown in [Waa1996] for INT that every (separable) complete metric space is weakly star-finitary. If we look at the proof, we see that it essentially uses only countable choice and the axiom **BDD** (see the appendix A.4.12), which is true in CLASS, INT and RUSS. This means that the theorem likely holds also in CLASS and RUSS. In CLASS 'weakly star-finitary' is equivalent to 'star-finitary'.

DEFINITION: Let $(\mathcal{V}, \mathcal{T}_{\#})$ be a spraid derived from $(V, \#, \preceq)$, where $\approx \subset V \times V$ is the complement of $\#$. We call $\approx$ and $(\mathcal{V}, \mathcal{T}_{\#})$ and $(V, \#, \preceq)$ *weakly star-finite* iff for each $a \in V$ the subset $\{b \in V \mid \lg(b) = \lg(a) \wedge b \approx a\}$ is subfinite, meaning there is $N \in \mathbb{N}$ such that $\{b \in V \mid \lg(b) = \lg(a) \wedge b \approx a\}$ contains at most N elements. A natural space $(\mathcal{W}, \mathcal{T}_{\#})$ is called *weakly star-finitary* iff $(\mathcal{W}, \mathcal{T}_{\#})$ is $\propto$-isomorphic to a weakly star-finite $\propto$-spraid. (END OF DEFINITION)

THEOREM: (in INT and likely CLASS and RUSS) Every complete metric space is homeomorphic to a weakly star-finite $\propto$-spread.

PROOF: (detailed sketch) The proof for INT is given in [Waa1996] (thm. 3.1.8) using $\mathbf{AC}_{10}$ in several lemmas. However, upon reflection one sees that each use of $\mathbf{AC}_{10}$ can be substituted by a use of **BDD** (see the appendix A.4.12), which is true in CLASS, INT and RUSS.

We sketch the translation to our setting in some detail. Let (X, d) be a complete metric space. We saw in (the proof of) theorem 3.4.3 that there is a $\propto$-spread $\mathcal{V}^{\wr}_{\mathrm{gd}}$ which is homeomorphic to (X, d). This spread $\mathcal{V}^{\wr}_{\mathrm{gd}}$ is derived from the neighborhood development of (X, d) formed by the collection $\{B(a_n, 2^{-s}) \mid n, s \in \mathbb{N}\}$ where $(a_n)_{n \in \mathbb{N}}$ is a dense sequence in (X, d). The strategy of the proof is to sharpen this neighborhood development into a strongly star-finite neighborhood development as follows.

By [Waa1996], thm. 3.1.1 (valid in BISH) a per-enumerable open cover of (X, d) has a strongly star-finite refinement.
We construct a strongly star-finite refinement $\mathcal{U}_0$ of $\{B(a_n, 2^{-0}) \mid n \in \mathbb{N}\}$. Then we construct a strongly star-finite refinement $\mathcal{U}_1$ of $\{B(a_n, 2^{-1}) \mid n \in \mathbb{N}\}$, and then of $\{B(a_n, 2^{-2}) \mid n \in \mathbb{N}\}$, etc. We obtain a sequence $(\mathcal{U}_n)_{n \in \mathbb{N}}$ of strongly star-finite covers such that for each U in $\mathcal{U}_n$: $diam(U) < 2^{-n}$. Now we use the elements of $\mathcal{U}_n$ as the $\lg(n)$ basic dots of our weakly star-finite spread $(\mathcal{W}, \mathcal{T}_{\#})$, where we need to be precise in defining $\preceq$, since these $\lg(n)$ dots belong to particular $\lg(n-1)$ dots. Here some real work needs to be done, also to define a weakly star-finite touch-relation $\approx$ on W such that for the corresponding apartness $\#$ we have: $(\mathcal{W}, \mathcal{T}_{\#})$ coincides with (X, d).

The entire proof, in order to be precise and correct, already for INT involves some more work than one might expect. We refer the reader to [Waa1996], and leave this proof-sketch for what it is: a sketch. (END OF PROOF)

4.0.10 Star-finite developments and 'efficient' metric spraids We turn once more to the APPLIED perspective. In paragraph 2.3.2 we showed that our representation of the real numbers has computational advantages regarding continuous $\mathbb{R}$-to-$\mathbb{R}$-functions. The general situation for complete metric spaces is more complicated. Yet very often we can find similar efficient representations for complete metric spaces. We do not delve into this in all detail, but we illustrate the basic idea.

DEFINITION: Given a metric space (X,d), let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a family of open (or neighborhood) covers where (i) $\mathcal{U}_n$ consists of inhabited sets of diameter less than 2^{-n} (ii) $\mathcal{U}_n$ is star-finite (iii) $\mathcal{U}_{n+1}$ is a (star-finite) refinement of $\mathcal{U}_n$ for each $n\in\mathbb{N}$. Then we call $(\mathcal{U}_n)_{n\in\mathbb{N}}$ a *regular star-finite development* of (X,d).
(END OF DEFINITION)

Suppose we have a complete metric space (X,d) and a regular star-finite development $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of (X,d). By the previous paragraph, this should not be a rare occurrence at all. Then we can build a spraid $(\mathcal{V},\mathcal{T}_\#)$ (with corresponding $(V,\#,\preceq)$) representing (X,d), where V represents $\bigcup_{n\in\mathbb{N}}\mathcal{U}_n$ (each basic dot corresponds to an element of some $\mathcal{U}_n$, except for $\bigcirc$). This is much more direct and efficient than the spread-representations of (X,d) given through theorems 1.2.3 and 3.4.3.

In this situation we would like a general way to replace a $\wr$-morphism from $(\mathcal{V},\mathcal{T}_\#)$ to another natural space $(\mathcal{W},\mathcal{T}_{\#_2})$ by an equivalent $\preceq$-morphism. We are confident that most important cases will resemble proposition 2.3.2. One direct example of this are of course the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^{\mathbb{N}}$. But we follow remark 2.3.2 a bit further to say something in general as well:

If $(\mathcal{W},\mathcal{T}_{\#_2})$ is $\preceq$-isomorphic to its 'finite-intersection-of-touching-basic-dots space', then we can follow the proof of proposition 2.3.2 closely to see that indeed a $\wr$-morphism from $(\mathcal{V},\mathcal{T}_\#)$ to $(\mathcal{W},\mathcal{T}_{\#_2})$ is represented by an equivalent $\preceq$-morphism.

We leave it to the interested reader to further ponder on the efficiency of different representations of complete metric spaces.

4.1 FROM NATURAL TO GENERAL TOPOLOGY

4.1.0 Invitation to general natural topology We think that from here one can develop a substantial part of general topology constructively, using pointwise concepts. We will however not tackle this significant undertaking in this monograph (which is lengthy enough as it is), preferring to invite others to do so. A very incomplete list of interesting subjects to cover/prove (some of which are also partly covered in [Waa1996]):

1. Metric topology and (in)finite-dimensional topology (see [vMil1989] for an excellent exposition of the latter)
   (a) Star-finite covers and partitions of unity
   (b) Topological and transitive notions of locatedness
   (c) Universal metric spaces and isometric (normed linear) extensions
   (d) Absolute retracts, Dugundji Theorem, Michael Selection Theorem
   (e) Compactification
   (f) Dimension theory and fractals
   (g) Topological manifolds
   (h) . . .
2. Algebraic topology, homotopy theory, . . .
3. Non-metric topology, topological lattice theory, separation axioms, . . .
4. Topological groups
5. Topological model theory
6. . . .

The list is probably easily expandable, since so little of classical general topology has been charted constructively. Apart from mathematics, this undertaking will have importance also for physics, we believe.

The relation to physics is one aspect that we wish to examine just a little bit more in this monograph, in our next and final section.

4.2 NATURAL TOPOLOGY AND PHYSICS

4.2.0 **A philosophical apology** This section is essentially philosophical in nature, although it employs mathematics as a vehicle. Its purpose is more to raise questions, than to give answers. The reader should ideally look at these questions with an open mind, to consider whether they merit more attention than usually given. Although some issues in this section are speculative to a certain degree, the author feels it would be a pity to ignore them.

4.2.1 **The strained relation between finiteness and infinity** One could argue that all constructive troubles regarding the foundations of mathematics stem from one strained relation: the one between finiteness and infinity. Brouwer's critique of CLASS can be summarized as: 'the notion in CLASS of infinity as a completed entity is fundamentally unsound'. According to Brouwer and other (pre-)intuitionists, we can only construct $\mathbb{N}$ as a 'potentially infinite' set, that is: growing in time beyond any fixed boundary, as time progresses (potentially) infinitely.

From the past century it becomes clear that the mathematical problem for 'infinitism' (denoted here as INFI) is to build a foundationally sound and satisfying theory. This has proved to be difficult mathematically, philosophically and not in the least socially because of the added difficulty to get mathematicians to agree on what is 'foundationally sound' and what is 'satisfying'.

Potential infinity is also contested by some, a position called (ultra)finitism (denoted here as FINI).[58] The mathematical problem for FINI is the same: to build a foundationally sound and satisfying theory. This has proved to be difficult also -apart from the social factor mentioned above- because of the mathematical and philosophical troubles that arise when trying to fix a good upper bound on $\mathbb{N}$.

It should be clear that the strain between INFI and FINI also has direct roots in physics. The 'simple' questions: 'What is time?' and 'Is the universe (fundamentally) finite?' have not been answered to any satisfactory degree, as far as we are aware.

It is no surprise therefore, that a lot of 'action' in the foundational debate takes place in the arena where we go from finiteness to infinity. For example,

[58] Well-known advocates are Alexander Esenin-Vol'pin and Edward Nelson.

each axiom used or discussed in this paper (see the axioms' section A.4 in the appendix) is tailored to describe certain transitions from finiteness to infinity or vice versa.[59]

And this transition is also precisely the thrust of 'natural spaces'.

CONVENTION: In the following (as in the previous...) we silently assume the world to be potentially infinite. The main reason to do so is that if the world is fundamentally finite, then philosophically speaking we can limit ourselves to discrete combinatorics, we think. So it seems more interesting for the debate to assume INFI, even though FINI makes as much sense. (END OF CONVENTION)

4.2.2 **Which mathematics are suited for physics?** Historically, it bears little surprise that classical mathematics (CLASS) is the mathematics of preference for physicists, which they generally use without question to develop physics. However, the author believes that such an unquestioning use of CLASS is difficult to support in the light of the developments in the foundations of mathematics.

As we have described in this monograph, there are several good reasons to doubt the inherent suitability of CLASS to describe our physical world. We do not mean to say that it is impossible to do physics well in CLASS, since any other approach can always be translated to a model in CLASS. But we are concerned about suitability, and possible blind spots arising from the exclusive use of CLASS. [60]

As stated, we believe that from our current state of knowledge in the foundations of mathematics one cannot simply choose one of the main alternatives CLASS, INT, or RUSS as the preferred mathematics for physics. One of the motivations for this paper is to show that natural topology yields a simple model in CLASS of basic principles in INT, and to couple this model with an explanation why we think that natural topology is better suited for physics than the standard practice.

[59] 'From here to eternity and back again'.

[60] As an example of what we believe to be unawareness of constructive foundational issues, one can take Hawking's recent book 'God created the Integers' ([Haw2005]) in which he discusses 31 great mathematicians of all times which according to him have been of fundamental importance to mathematics and physics. Brouwer is not among them, nor is he even referenced, even though both his topological work and the foundational crisis that he discovered have had a major impact.

Yet as we have seen, this leaves unresolved another fundamental aspect of the relation between physics and mathematics, namely the question what reason we have to assume that reality is modeled better by a compact unit interval than by a non-compact unit interval. Or by paradigm leap: what reason do we have to assume that Nature is capable of producing non-recursive real numbers? A strong case can be made, we believe, for a form of Laplacian determinism in which Nature can only produce recursive sequences.[61] This belief is often written as $\mathbf{CT}_{\mathrm{phys}}$, where **CT** stands for Church's Thesis: 'every infinite sequence of natural numbers is given by a recursive algorithm'.

This question is debated in some circles, yet in the author's eyes the debate mostly lacks mathematical precision. In [Waa2005], a precise mathematical setting is given in which $\mathbf{CT}_{\mathrm{phys}}$ can even be experimentally tested, although the latter involves some serious computational effort and has not yet been done. In addition there is no guarantee of result for this experiment. We will comment on this further on, because even as a thought-experiment alone, doubt is cast on the intrinsic validity of the standard probability approach in science. Laplace himself already raised similar doubts but resolved them using a model of 'limited information' ([Lap1776], also see [Lap1814]):

> "*Before going further, it is important to pin down the sense of the words chance and probability. We look upon a thing as the effect of chance when we see nothing regular in it, nothing that manifests design, and when furthermore we are ignorant of the causes that brought it about. Thus, chance has no reality in itself. It is nothing but a term for expressing our ignorance of the way in which the various aspects of a phenomenon are interconnected and related to the rest of nature.*"

A different model of 'limited information' will also serve us well.

Apart from foundational probability issues, in [Waa2005], the open question is raised whether RUSS could be a better model for physics than INT or CLASS. What remains lacking in the literature (as far as we are aware) is a sharp analysis of the possible consequences of $\mathbf{CT}_{\mathrm{phys}}$ for physics.

REMARK: In [Waa2005] a possible partial reconciliation between RUSS and INT was left uninvestigated. We will describe such a partial reconciliation in this section, using a model of limited information which resembles 'type two effectivity' (TTE). This partial reconciliation is relevant for physics, we believe,

[61] A similar belief seems to underly Stephen Wolfram's 'A New Kind of Science' ([Wol2002]), but without a sharp foundational motivation that we know of.

in the following sense. Suppose that $\mathbf{CT}_{\mathrm{phys}}$ is one day seen to hold.[62] Then the reconciliation that we propose below offers an explanation why the physical world would still (appear to) conform to **BT**. (END OF REMARK)

4.2.3 Kleene's function realizability and type two effectivity One of the developments with some similarity to natural spaces is Weihrauch's type two effectivity.[63] TTE uses classical logic, and can therefore in our eyes not be considered a constructive theory, yet there is a certain conceptual overlap we believe.

Kleene's function realizability used in a setting similar to TTE may give us a constructive way to formally interpret INT as part of RUSS. We leave this as an open question.[64]

The basic idea of TTE is not too difficult. One considers Turing machines which can handle infinite input tapes, and which produce infinite output tapes step-by-step (like Brouwer's choice sequences). In other words, a given machine T_e is fed an infinite input sequence α (not necessarily recursive!) and it is possible that after some finite time T_e computes some first result from an initial segment of α, which is then the first output value of a (possibly) infinite output sequence $\beta = T_e(\alpha)$. We refer the reader to the literature, notably [Wei2000].

Since we wish to limit the scope of this article to our own limited level of knowledge, we will not delve into TTE and Kleene realizability. Instead we present a simple two-player game which we believe gives a similar (but informal) model of INT in RUSS.

4.2.4 An informal two-player game representing INT in RUSS We describe a game called 'Limited Information for Earthlings' (LIfE). We start out by introducing our two players, and describing their rules of conduct.

Player I is called 'Giver of Digits' (GoD), and player II is called 'Humble Mathematician accepting Numbers' (HuMaN).

The basic idea of the game LIfE is that GoD gives infinite sequences of numbers to HuMaN where all sequences are in fact given by a recursive algorithm, but HuMaN does not know this. We list the rules of LIfE:

[62] For instance by a positive outcome to the experiment described in [Waa2005].

[63] TTE, to which of course many others have also contributed, see the literature.

[64] Which most likely has been answered already outside the author's awareness, see also paragraph A.0.0.

L$_1$ GoD has at her disposal all infinite recursive sequences (all $\alpha \in \mathbb{N}^{\mathbb{N}}$ for which there is a Turing algorithm which computes α).[65] GoD hands out such sequences α to HuMaN, but is allowed to do so step-by-step without disclosing any information about the algorithm computing α.

L$_2$ GoD may lie in two different ways (and does). First by telling HuMaN that not all sequences are recursive. Secondly, GoD is allowed to cheat with information on the sequences and algorithms involved, if HuMaN cannot know the difference. For example, if at stage n GoD has given out only the finitely many values $\overline{\alpha}(n)$ for a certain computable α, then GoD may switch to any computable β with $\overline{\beta}(n) = \overline{\alpha}(n)$. However, during the game and when playing a sequence, GoD may only switch a finite number of times. Compliance to this can be determined post-game, HuMaN remains ignorant.

On the other hand, GoD also can disclose part or all of the algorithm computing α. This must of course be done consistently, so GoD can only switch to other sequences conforming to earlier disclosures.

L$_3$ The player HuMaN has to build mathematics from the sequences given by GoD, but HuMaN can also at any time use his own recursively defined sequences. HuMaN believes that GoD also has non-recursive sequences at her disposal (although this is in fact false). GoD is obliged to give to HuMaN as much initial values of a sequence α as HuMaN wants. (But GoD is allowed to cheat by **L**$_2$).

L$_4$ HuMaN can only use constructive logic, implying for instance that existential statements and disjunction (A or B) must have constructive content.

So what are the truths of LIfE for HuMaN? We think that for HuMaN, from our perspective, LIfE is a model of Brouwer's intuitionism. The sequences that form HuMaN's mathematical universe correspond to the universal spread σ_ω, or natural Baire space through Brouwer's eyes. Brouwer's motivation for the Continuity Principle **CP** transfers directly to our game LIfE (we can even prove it, see below). The same holds for **BT**, since we see no other way for HuMaN to know that a certain $B \subset \mathbb{N}^*$ is a bar on $\mathcal{N}$ other than by having constructed a genetic bar on $\mathcal{N}$ from which B descends. (We can prove that if GoD is omniscient, then $\neg\, \exists B \subset \mathbb{N}^* [B$ is a non-inductive bar on $\mathcal{N}]$ holds in LIfE.)

[65] This is equivalent to: $\exists e \in \mathbb{N} \forall n \in \mathbb{N} \exists k \in \mathbb{N}\, [T(e, n, k) \wedge \mathrm{Outc}(k) = \alpha(n)]$, if we use Kleene's T-predicate.

Therefore, we think that the truths of LIfE for HuMaN (seen from our perspective) are precisely the theorems of INT. But from GoD's perspective, everything in LIfE is algorithmically determined, in other words LIfE is part of RUSS.

REMARK: The situation in which from an omniscient perspective sequences are recursively determined, whereas from the limited-information perspective the mathematical truths are intuitionistic, might provide -should the need arise[66]- a way to explain that nature 'behaves inductively', so to speak.

Or to put it differently: even if one day $\mathbf{CT}_{\text{phys}}$ is believed to hold, then we will be unlikely to encounter a Kleene Tree in nature, because nature itself is most likely built inductively. The experiment in [Waa2005] however indicates a (perhaps practically infeasible) way to expose the statistical anomalies which would result from $\mathbf{CT}_{\text{phys}}$. This corresponds to a strange truth in LIfE: if HuMaN is given enough time, HuMaN can discover that GoD is lying about having non-recursive sequences, by using the Kleene Tree and our standard model of probability for the testing of physics' hypotheses.

Yet the real rub if one believes $\mathbf{CT}_{\text{phys}}$, is that our standard model of probability used to prove physics' theories would seem to need serious revision. Ironically, this model is largely due to Laplace. He could however point out, in accordance to the quote above (4.2.2), that knowledge of the truth of $\mathbf{CT}_{\text{phys}}$ alters the state of our ignorance, and therefore by necessity also our probability models. (END OF REMARK)

We now formulate our last meta-theorem concerning the game LIfE. It should be seen as an alternative illustration of our attempts to further the 'reunion of the antipodes' [67] in constructive mathematics, and not as an important result in itself.

META-THEOREM:

(i) In LIfE, we can prove **CP**.

(ii) Suppose GoD is omniscient. Then we can prove $\neg\exists B \subset \mathbb{N}^*$ [B is a non-inductive bar on $\mathcal{N}$] for LIfE.

(iii) Given enough time, HuMaN can discover that by an overwhelming odds ratio, GoD plays only recursive sequences.

PROOF: We prove this metatheorem in the appendix A.3.19. (END OF PROOF)

[66] Which probably occurs, we believe, if one day $\mathbf{CT}_{\text{phys}}$ should be validated.

[67] Derived from Schuster's (et al.) terminology 'reuniting the antipodes', see [Schu2001].

COROLLARY: In CLASS, we can prove **CP** and **BT** for the game LIfE. (This would seem to express the situation in TTE).

EXAMPLE: We show that, assuming GoD's omniscience, the Kleene Tree (K_{bar}, see 2.5.3) is not a bar on σ_2 in LIfE even though all σ_2-sequences in LIfE are in fact recursive. For suppose HuMaN proves that a subset $B \subset \{0,1\}^*$ is a bar such that $K_{\text{bar}} \subseteq B$. Then GoD, with omniscience, can choose a recursive sequence α such that $\{\overline{\alpha}(i)\}_{\preceq} \cap K_{\text{bar}}$ is infinite for any $i \leq N$, where N is the index such that $\overline{\alpha}(N) \in B$. (GoD plays a β step-by-step and at each step determines whether $\overline{\beta}(n) \star 0 \cap K_{\text{bar}}$ is infinite, if so then the next choice for β is a 0, if not then the next choice is a 1. At a certain point in time M, HuMaN must produce N such that $\overline{\beta}(N) \in B$. At this point, GoD fixes $\alpha = \overline{\beta}(N) \star \underline{0}$, which is a recursive sequence. We see that GoD has played by the rules.).

By the properties of α, we see that $\overline{\alpha}(N) \in B, \overline{\alpha}(N) \notin K_{\text{bar}}$, in other words $B \neq K_{\text{bar}}$.

APPENDIX

Additionals, Examples, Proofs

The appendix starts with closing remarks and personal acknowledgements. Then we give some motivation and historical background of this paper, and a small recap. We add a brief history of apartness topology.

The appendix is predominantly taken up by examples and proofs. The examples are worked out in some (but not all) detail, leaving some aspects as exercises to the reader. The proofs are worked out in almost all details, yet occasionally we also leave some exercise to the reader.

To be foundationally precise (although more precision is certainly possible) we present and explain the axioms used and discussed in this monograph.

We end with the bibliography.

A.0 CLOSING REMARKS AND ACKNOWLEDGEMENTS

A.0.0 Some closing remarks Our introductory exposition of natural topology ends here (bar the appendix). Although the author would have liked to be able to present more, it proved to be a truly energy-consuming undertaking to work out, in an elegant way, the precise machinery needed for natural topology.
Not having been employed as a mathematician for the past 15 years, there is a limit to how much energy one can reasonably spend on such undertakings. However, this particular project is worthwhile for the author personally, since it has answered some questions which kept on resurfacing after writing his PhD-thesis in 1996.

We hope that others will also see some merit in this project. Experience teaches that new concepts take time and effort to get used to, and the author is no exception. The effort which went into writing this study has precluded him from really studying various promising other concepts, such as Abstract Stone Duality (see [Tay&Bau2009]) and formal topology (see e.g. [Sam2003]). Also, any real comparative study should be done from a deeper understanding of various related fields than the author possesses. He apologizes for his omissions-by-ignorance and inaccuracies on this count.

Reactions to this study are most welcome. Please do not hesitate to point out errors, omissions, other points of view, interesting sources for an update of the bibliography, solutions to questions, results of this paper which were already proved elsewhere, etc.
On this last item, we should state that it is likely that some of our results resemble or even equal results proved elsewhere. Not mentioning a source in this respect only shows that the author is unaware of such a source, which he will be happy to name when pointed out.
Finally, the appendix contains a ‘background and motivation’ section with a short historical overview. This overview is likely to contain inaccuracies, and suggestions for improvement are most welcome.

A.0.1 Acknowledgements of the author First and foremost, I would like to thank Wim Couwenberg for his continuing interest and active participation is this seemingly unending project. But what a beautiful project it is, in my humble opinion, and it was sparked off by Wim’s insight and insistence that a

simple presentation of ‘vlekjesruimten’ (‘dotty spaces’...which for marketing reasons we have renamed natural spaces...) should be possible. Wim’s basic idea was perhaps not so simple to develop as we had hoped, but now allows for a nice and faithful classical interpretation of many results in intuitionistic topology.
So the credit for the concept of ‘natural space’ goes to Wim, but more importantly his support and friendship have kept this project afloat. There were several times when I was convinced that practically nobody is interested in intuitionism, and that in natural spaces there was little worth mentioning when compared to formal topology and other disciplines.
Wim convinced me otherwise, by repeating his questions about intuitionism and other constructive disciplines and by becoming captivated by the constructive approach and foundations in general. Wim underlined the need for a simple exposition of intuitionistic ideas for classical mathematicians, and convinced me that this monograph fulfills a worthwhile role in that respect. He also actively participated in thrashing out the first and most illuminating versions of ‘natural space’ and ‘natural morphism’. His sparring role in many other discussions has simply been invaluable.

It turned out however to be still a daunting task to get down to the nitty-gritty and have everything in precise mathematical working order. Since this nitty-gritty lies close to my PhD thesis, and the comparative study of constructive foundations done in [Waa2005], it turned out that I had better write this monograph in solo fashion.[1] We decided this after a joint talk last January, which revealed that the foundational complexities surrounding ‘natural spaces’ could not be disregarded. I can only hope that with this text I have done justice to all our discussions (especially the fun of discovering yet another snag!) and working together.

Second, I would like to dedicate this paper to Wim Veldman, who introduced me to foundations, intuitionism and constructive mathematics, and was my doctoral advisor during my PhD research. Wim Veldman’s precise and elegant style reflects on his insistence that a structural framework for constructive mathematics should be both elegant and foundationally precise. I hope that this paper passes muster in that respect. Wim’s active work in intuitionism is also an inspiration, I took just one of his results as an important illustration in this text.

Furthermore there are many people, too many to list individually, who have contributed in some way or other to the epigenesis of this paper. I would like

[1](Final) definitions, results, proofs, examples, and errors therefore are the author’s.

to mention Bas Spitters who is always willing to discuss and explain formal topology beyond my limited knowledge and understanding. Some twelve years ago, Peter Schuster (et al.) already contributed his apt terminology 'reuniting the antipodes', which has been a continuing inspiration. Giovanni Sambin wrote me some very kind words when I felt miserable for mistakenly having claimed to have spotted an error in a formal-topology paper. The tireless work of Douglas Bridges to carry on the program started by Bishop has also played an important part. It is my hope that this monograph will help to continue investigating 'constructive mathematics' in the spirit of Bishop, by showing how to inductivize pointwise notions.

Perhaps one day the difference between Brouwer's and Bishop's approach will be felt to be far less important than their correspondence. Studying constructive mathematics in the pointwise style of classical mathematics seems to me to remain the most attractive choice. I do however feel that a transparent axiomatization similar to FIM is always called for, even though it shouldn't have to dominate the presentation. This is a matter of taste also, but recent work in constructive reverse mathematics by various authors (notably Ishihara, see e.g. [Ish2006], and Veldman, see e.g. [Vel2011]) has shown the benefits of further explicitizing relative axiomatic dependencies.[2]

The support from my family and my friends, although mostly in other areas than mathematics, has played a major role in this project. Finally, the love of my wife Suzan and my daughters Nora and Femke (not to mention their patience with this project), has been quite indispensable.

I would like to thank all these people mentioned above very sincerely.

(the author, july 2011)

A.0.2 **Hommage to Brouwer, Kleene, Bishop and...** One may also see this paper as a hommage to Brouwer, Kleene and Bishop, but let us not forget all those other mathematicians who have worked hard to both expand and simplify mathematics. This is an ongoing collective endeavour, notwithstanding differences of style, character and opinion. The bibliography, incomplete as it may be, should serve to illustrate just how many people are dedicated to developing constructive mathematics. Behind each name and reference lies a body of related work which is left unmentioned, but with the help of the internet should be easily findable.

[2] Veldman has also developed 'Basic Intuitionistic Mathematics' (BIM) as a formalization comparable to FIM in which reverse intuitionistic mathematics can be carried out.

A.1 BACKGROUND, MOTIVATION, AND RECAP

A.1.0 Background Much of theoretical mathematics is built on idealizations which fail in real life. A simple example is that of a floating point representation of a real number x very close to 0, where the decision whether $x = 0$ or $x \neq 0$ can be needed for further computations. Theoretically this decision is trivial, but in practice we cannot always determine whether $x = 0$ or $x \neq 0$. That is one situation where 'applied mathematics' comes in, with mathematicians and computer scientists working to translate theoretical mathematical ideas to 'real life' situations.

One often comes across excellent practical solutions, which are yet ad hoc in character. People working in the field of applied mathematics do not seem to consider this a problem, but we find it interesting to note that a more coherent and unified approach is possible. This might hopefully shed some new light on theoretical science as well, and help explain some important ideas of intuitionism and constructive mathematics to the 'working' mathematician who is used to classical mathematics.

This paper is concerned with such an approach, from the standpoint of topology. A lot of work towards bridging the gap between theoretical and practical mathematics has been done in what is known as constructive mathematics. Although constructive mathematics in its essence is as old as mathematics itself, one can still consider Brouwer to be its founding father. Brouwer was the first to critically analyze the body of classical mathematics to come to the conclusion that the principle of the excluded middle (**PEM**) could only be maintained for infinite sets at the cost of constructivity. Brouwer sharply demonstrated this mathematically, with various clever examples, showing e.g. that for an arbitrary real-valued continuous function f on the real interval $[0, 1]$ one cannot always construct an x in $[0, 1]$ where f assumes a maximum. By the very nature of his critique, the foundations of mathematics in general, and especially the then popular and newly evolving mathematical discipline of set theory (started by Cantor) were shaken badly. It must be said that already Kronecker and Poincaré had serious reservations about Cantor's treatment of infinity, and also among his contemporaries Brouwer was not the only one with constructive views.

However Brouwer, the master topologist, did not content himself with criticism alone. From 1912 to roughly 1927 he developed a new constructive framework for mathematics, called intuitionism. One of his insights was that an everywhere-defined real function (to the reals) has to be continuous (we rediscover the reason for this in this paper). But he also encountered some difficult obstacles in building this constructive framework. He presented his solutions to these obstacles as theorems, albeit with rather unorthodox proofs, in none too easy language to the mathematical community of his time. This mathematical community was mostly unreceptive to Brouwer's critique of the classical foundations, and unwilling to change its comfortable views on classical mathematics as being the only viable framework for doing math. Hilbert (who in 1928 ousted Brouwer from the board of the Mathematische Annalen, showing how deep his resentment of Brouwer's views had become) is famously quoted to have said: 'No one shall expel us from the paradise that Cantor created for us'.

Deeply disappointed, Brouwer more or less retreated in his own activities and never regained his former prolific productivity. Still, Brouwer had followers, notably his student Heyting who simplified and formalized his mentor's intuitionism to make it more accessible. Later on, the computability expert Kleene became a supporter of Brouwer's ideas. Kleene managed to axiomatize intuitionism in a very clear way in [Kle&Ves1965], to prove relative consistency (through realizability) which further opened the door for interested mathematicians. Kleene discovered that under these axioms Brouwer's notion of 'choice sequence' could not coincide with the notion of 'computable sequence', a result which will also concern us in this paper since the 'true' reason for this is topological in nature. Namely in recursive mathematics the Cantor space $\{0,1\}^{\mathbb{N}}$ is not compact. Kleene thus also showed that Brouwer's Fan Theorem (**FT**) (stating compactness of the Cantor space $\{0,1\}^{\mathbb{N}}$) was truly an axiom, in the sense that it could not be proved from the other axioms.

Around that time, Bishop also became convinced of the intrinsic worth of the constructive framework. However, Bishop was not attracted to the foundational discussions, involving a great deal of logic, logicism and axiomatics. Bishop wanted to build a solid body of constructive mathematics, picking up where Brouwer had left around 1930, and also in such a way that the results would be acceptable to classical mathematicians as well. By this time, computers had already entered the scene, and mathematical awareness of computability and computational issues had increased the reception of constructivism. Bishop-style mathematics (BISH) has increasingly become popular, not in the least because of its refocussing on 'plain mathematics' instead of logic and foundations, and its down-to-earth approach. This approach started with a comprehensive treatise of constructive analysis, in the context of metric spaces ([Bis1967], [Bis&Bri1985]). Bishop stated that there was little need for general topology, and that 'mystic' axioms like the Fan Theorem (although classically true) were unnecessary if one chose the right definitions. However, it was later shown ([Waa1996], [Waa2005]) that Bishop's definitions practically imply the Fan Theorem. Bishop's work is carried on by many, including notably Bridges. Brouwer's intuitionism seems to attract less mathematical attention, yet is carried on notably by Veldman.

From the 1960's on, a parallel development of many authors (Scott, Martin-Löf, Fourman, Sambin, et al.) led to the field of domain theory, pointfree topology and formal topology. Actually already Freudenthal started with (intuitionistic) pointfree topology in [Fre1937], reacting to Brouwer and using ideas of Alexandroff and Hurewicz. In recent developments in formal topology, a number of the above issues have been dealt with in such a way that one can view this as a reunion of parts of the different approaches. This topological setting is no coincidence, since Brouwer was a brilliant topologist, and Brouwer's intuitionism was built with the backing of his topological expertise. Bishop seems to have underestimated the topological necessity of Brouwer's Fan Theorem in order to build a constructive model of analysis in which continuous functions on compact spaces are uniformly continuous.

A.1.1 Motivation and some results The accomplishments of formal topology notwithstanding, we feel that formal topology in many of its current presentations (a growing number of papers and tutorials in a formal categorical style) lacks the intuitive appeal of both Brouwer and especially Bishop. Much of this is due to the fact that both Brouwer and Bishop concentrated on 'points' and 'spaces' in the usual mathematical way, and limited themselves to some form of separability (enumerable bases, enumerable dense subsets) in order to achieve constructivity. Also, there are still some foundational concerns surrounding compactness which we feel merit attention.[3]

Interestingly enough, such a separability approach is ideally suited for a simple and elegant version of pointfree topology which we think deserves the name 'natural topology' for three reasons. First of all, we think that natural

[3] An alternative approach called Abstract Stone Duality (ASD) was recently developed by Taylor and Bauer, see also [Tay&Bau2009]. This development seems partly motivated by a search similar to ours for a simpler and more directly appealing approach to constructive topology. However, ASD draws on many areas in which the author is not proficient, and therefore comparison here is not possible. Kalantari and Welch (see e.g. [Kal&Wel2006]) have also been developing related concepts in a recursive-computable framework.

topology is ideally suited for dealing with the study of nature (in other words physics...), since natural phenomena can only be observed and measured by scientists in a manner corresponding to these definitions. Secondly, in natural topology one builds a space of points in a natural way, and immediately sees a topology on this space arising from the construction, matching the space. Thirdly, we believe this allows for a natural pointwise style of mathematics, now that the most foundational aspects have been established.[4]

Another advantage of this approach we hope to show in this paper: the easier translation of existing (intuitionistic) results to this setting. One such result is that all complete separable metric spaces are representable as a natural space. Another is that all natural spaces arise as a quotient space of Baire space, something which Brouwer already incorporated into his theory of spreads. Still another important translated result is the Heine-Borel property for inductive covers of subfanns ($\mathbf{HB}_{\propto}$, see 3.2.1). This relates to (and generalizes we believe) the result in [Neg&Ced1996] concerning the Heine-Borel property for (formal-topological) inductive covers of the formal real interval $[\alpha, \beta]$. As an icing to the cake we prove (in BISH) the metrizability of star-finitary natural spaces. The corresponding intuitionistic result from [Waa1996] has not been transposed to a formal-topological setting (as far as we are aware). The constructive theorem seems comparable to the classical metrization results of strongly paracompact spaces.

Since we can define natural spreads and fans (and even spraids and fanns), we are led to the meta-theorem that natural Baire spreads (fans) correspond precisely to Brouwer's spreads (fans). Moreover, natural morphisms correspond precisely to Brouwer's spread-functions. Natural morphisms are suited, we think, for computational purposes also.[5] The definition of spraids and fanns is made to facilitate both computational practice and (topological) lattice theory. For nice papers on computation and constructive topology see [GNSW2007], which also contains a large list of references and detailed historical background, and [BauKav2009]. In [BauKav2009], recommendations are made for efficient exact real arithmetic, which seem to match our definition of $\sigma_{\mathbb{R}}$ and $\preceq$-morphisms quite closely.

A.1.2 To recap In this monograph we develop a simple topological framework, which can serve as a theoretical yet applicable basis for dealing with real-world phenomena. The paper is self-contained, but some familiarity with

[4] This monograph needs to focus on foundations also, which makes for less easy reading.

[5] In formal topology mappings are usually defined as multivalued relations, which seems computationally more complicated, but the author's knowledge on this is again limited.

basic topology is probably necessary for understanding its build-up. It turns out that this approach covers not only the real numbers, but in fact all 'separable spaces', meaning topological spaces having an enumerable dense subset (for $\mathbb{R}$ consider e.g. $\mathbb{Q}$).

In the first half of the paper we present the framework and formulate the basic theorems and properties. We discuss the strong connection with applied mathematics and physics, and give examples of computational and topological issues in applied mathematics. When developing the theory further, we discover some foundational issues surrounding compactness, leading also to questions on the topological character of our physical universe.

The second half of the paper is therefore more foundational in nature, exploring possible avenues to resolve these issues. Links with existing frameworks and theories are discussed, especially classical mathematics (CLASS), recursive mathematics (RUSS), intuitionism (INT), Bishop-style mathematics (BISH), and formal topology. Natural topology can also be seen as a simplified version of formal topology, or of domain theory. The simplification fits in the historical line of simplification efforts of Heyting, Freudenthal, Kleene, Scott, Bishop, Martin-Löf, Bridges, Veldman[6] and many others (see also Troelstra's and van Dalen's standard treatise on constructive mathematics [Tro&vDal1988]).

In the final section, we return to discuss the relation with physics. We think that the question of which mathematics to choose for physics deserves more attention. We believe that natural spaces provide a strong conceptual reference frame for physics. Compactness issues also play a role here, since the question 'can Nature produce a non-recursive sequence?' finds a negative answer in $\mathbf{CT}_{\text{phys}}$. $\mathbf{CT}_{\text{phys}}$, if true, would seem at first glance to point to RUSS as the mathematics of choice for physics. To discuss this issue, we wax more philosophical. We present a simple informal model of INT within RUSS, in a two-player game called 'Limited Information for Earthlings' (LIfE) with players 'Giver of Digits' (GoD) and 'Humble Mathematician accepting Numbers' (HuMaN). We also point to [Waa2005] for a physical experiment which could cast light on $\mathbf{CT}_{\text{phys}}$.

In the appendix we work out more interesting details regarding the examples given in the first half of the paper. One of these examples is the line-calling decision-support system Hawk-Eye, for which we recommend a LET feature. The appendix also contains most of the proofs.

[6] Wim Veldman's lecture notes [Vel1985] (in Dutch) are a very nice exposition of intuitionistic mathematics, one hopes for an English translation some day.

We hope that this monograph will serve as a welcoming introduction to the varieties of (constructive) topology. Our aim is to compare these varieties in such a way that their presentation is simplified and their mutual differences are reduced to their essence. One result that we are happy to mention is that the presented framework of natural spaces gives a faithful classical representation of basic intuitionistic results.

Looking at the results in this monograph, the author comes to the conclusion that Brouwer's concepts and axioms are still of exceptional elegance and relevance for constructive mathematics. In fact INT is the only constructive theory we know with an elegant pointwise approach which solves all the compactness issues that we studied in chapter three. Perhaps even more relevant: the axioms of INT are precise and appeal directly to the author's mathematical intuition. This monograph should therefore also be seen as strongly supportive of further development of INT.

A.1.3 Brief historical note on apartness topology We can say that apartness topology started with Brouwer, and was given a pointfree flavour by Freudenthal in [Fre1937]. This since intuitionistic topology practically entails all the phenomena of apartness topology. Troelstra also studied intuitionistic topology in [Tro1966]. Martin-Löf provided a strong germ for constructive pointfree topology in [M-Löf1970] (also see A.1.0). In [Coq1996], Coquand even asks the question whether starting formal topology from an inequality relation (apartness) would be worthwhile. Kalantari and Welch (see e.g. [Kal&Wel2006]) have been developing related concepts in a recursive-computable framework.

As far as we can tell, the first definition of 'apartness topology' and 'apartness space' was given in [Waa1996], a study of modern intuitionistic topology which was set up in such a way as to attract attention also from people in Bishop's school. Apartness topology plays a central role in [Waa1996], just as in this account. The definition in [Waa1996] was given in BISH, but with the use of **CP** in mind to arrive at basically the same topology as the natural topology of the current paper.[7]

Some years later, Bridges and Vîţă developed a related but different notion, also called apartness topology (or apartness spaces), which does not depend

[7] The idea of 'apartness topology' came from an earlier study of intuitionistic model theory (see [Vel&Waa1996]), where the current author discovered that first-order sentences using only apartness can describe topological properties, due to the presence of **CP**. This gives a correspondence between intuitionistic model theory and classical topological model theory.

on the use of **CP** (see [Bri&Vîţ2011]). There have been a number of articles by different authors on this different notion of apartness topology since then. In [Bri&Vîţ2009] there is also a treatment of lattices and pointfree machinery, but it seems different from the treatment here.

These articles do not, as far as we are aware, address the question whether we can find interesting apartness spaces which are not already equivalent to a metric space. As we show in this paper, non-metrizable apartness spaces arise naturally in the context of infinite-dimensional topology, an area where constructive methods should be fruitful but which has been somewhat lagging behind in constructive topological investigations, we believe.

The above should illustrate that the main ideas in this monograph can already be found in older sources. What natural topology has to offer, is a new combination of these ideas (with a high level of detail). Much of classical separable topology (and mathematics) is still uncharted from a constructive perspective. There is in other words yet a nice long road ahead of us.

A.2 EXAMPLES

A.2.0 Hawk-Eye (See 1.3.0:) The line-calling decision-support system Hawk-Eye was critically analyzed by Collins and Evans in [Col&Eva2008], in which they raise a concern which resembles our claim (repeated below) and many other concerns about error margins and measuring. However, the precise nature of the problem associated to what they call ‘digitizing’ (making a decision based on continuous data) is left undiscussed. We therefore state explicitly:

claim Hawk-Eye, irrespective of the precision of the cameras, will systematically call OUT certain balls which are measurably IN or vice versa.

To see why this is so, it is enough to notice that any real measurement and calculation derived from this measurement, resulting in either IN or OUT, correspond to a function from $\mathrm{R}^*_{\mathbb{Q}}$ to $\{\mathrm{in}, \mathrm{out}\}$. For simplicity’s sake let’s put the border of a line at the natural real number $0 \in \mathbb{R}_{\mathrm{nat}}$, where a given trajectory end x being IN corresponds model-wise to $x \geq 0$. To give Hawk-Eye credit, we will assume that trajectories are calculated mathematically correctly from data entered. Now the situation for a ball to just touch or just not touch the line can be translated by looking at a shrinking sequence of rational intervals hovering around 0.

This means that Hawk-Eye, for each such shrinking sequence, must yield either IN or OUT, after only a finite number of intervals in the sequence (a Wimbledon match must be finished before August, say). Taking the shrinking sequence $x = [-1, 1], [-\frac{1}{2}, \frac{1}{2}], [-\frac{1}{4}, \frac{1}{4}] \ldots$ we determine Hawk-Eye’s decision on x, say IN, which is arrived on say at interval $[-2^{-m}, 2^{-m}]$. Clearly then there are balls whose translated trajectory starts out with $[-1, 1], [-\frac{1}{2}, \frac{1}{2}], [-\frac{1}{4}, \frac{1}{4}], \ldots, [-2^{-m}, 2^{-m}]$, which are nonetheless OUT by a margin of 2^{-m-1} (surely measurable, if cameras of Hawk-Eye can measure up to $[-2^{-m}, 2^{-m}]$).

The claim is not per se important. Hawk-Eye admits to an inaccuracy of around 3 mm.[8] This is usually blamed on inaccuracy of the camera system. But regardless of camera precision we cannot expect to solve the topological problem that there is no natural morphism from the real numbers to a two-point space $\{\mathrm{IN}, \mathrm{OUT}\}$ which takes both values IN and OUT. To make this precise we define:

[8] Still, one sees ‘sure’ decisions being pronounced by the system where the margin is smaller. A Nadal-Federer match in which this occurred for a margin of 1 mm attracted some media attention to Hawk-Eye’s (in)accuracy.

DEFINITION: For $m, n \in \mathbb{N}$ we write $\overline{\underline{m}}(n)$ for the sequence $m, \ldots, m$ of length n. Let $T_{2_{\text{nat}}} = \{\overline{\underline{0}}(n) \mid n \in \mathbb{N}\} \cup \{\overline{\underline{1}}(n) \mid n \in \mathbb{N}\} = \{0\}^* \cup \{1\}^*$. Similarly, let $T_{3_{\text{nat}}} = \{0\}^* \cup \{1\}^* \cup \{2\}^*$. Put $\overline{\underline{2}}(s) \# \overline{\underline{0}}(m) \# \overline{\underline{1}}(n) \# \overline{\underline{2}}(s)$ for all $n, m, s > 0$. Likewise put $\overline{\underline{0}}(m) \preceq \overline{\underline{0}}(n)$, $\overline{\underline{1}}(m) \preceq \overline{\underline{1}}(n)$ and $\overline{\underline{2}}(m) \preceq \overline{\underline{2}}(n)$ for all $m \geq n$. Then $(T_{2_{\text{nat}}}, \#, \preceq)$ and $(T_{3_{\text{nat}}}, \#, \preceq)$ are pre-natural spaces with as maximal dot the empty sequence of length 0. Write $(2_{\text{nat}}, \mathcal{T}_{2_{\text{nat}}})$ and $(3_{\text{nat}}, \mathcal{T}_{3_{\text{nat}}})$ for the corresponding natural spaces, which up to equivalence contain precisely two (resp. three) points $\underline{0} = 0, 00, 000, \ldots$ and $\underline{1} = 1, 11, 111, \ldots$ (and resp. $\underline{2} = 2, 22, 222, \ldots$). (END OF DEFINITION)

From theorem 1.1.2 it follows directly that any natural morphism f from $\mathbb{R}_{\text{nat}}$ to 2_{nat} is constant (meaning either $f(x) \equiv \underline{0}$ for all x in $\mathbb{R}_{\text{nat}}$ or $f(x) \equiv \underline{1}$ for all x in $\mathbb{R}_{\text{nat}}$). So there is no surjective natural morphism from $\mathbb{R}_{\text{nat}}$ to 2_{nat}.

REMARK: This is not the end of the line though for Hawk-Eye-like applications. One restriction on morphisms can and should be relaxed in cases like Hawk-Eye, namely the restriction that morphisms respect the apartness relation $\#_{\mathbb{R}}$. By this we mean that we should turn to morphisms on the unglueing $\sigma_{\mathbb{R}}^{\between}$ of $\sigma_{\mathbb{R}}$, where $\sigma_{\mathbb{R}}^{\between}$ is equipped with the finer apartness $\#_\omega$ given by $a \#_\omega b$ iff $(a \not\preceq b \wedge b \not\preceq a)$.

If we turn to the natural space $\mathbb{R}_{\#_\omega}$ derived from $(\sigma_{\mathbb{R}}^{\between}, \#_\omega, \preceq)$, then we see that there are many surjective morphisms from $\mathbb{R}_{\#_\omega}$ to 2_{nat}. This means that we can for example represent the situation $\forall x \in \mathbb{R}[x > 0 \vee x < 1]$ by a morphism h from $\mathbb{R}_{\#_\omega}$ to 2_{nat} such that $h(x) = 0$ implies $x > 0$ and $h(x) = 1$ implies $x < 1$. (END OF REMARK)

So finally, how should Hawk-Eye be amended? Clearly our recommendation to Hawk-Eye is to introduce a LET-feature. Suppose for simplicity that Hawk-Eye's camera-cum-software margin of error can safely be taken to be 2.5 mm $\approx 2^{-8}$ m.[9] Since Hawk-Eye calculates the trajectory end x of a ball from several camera measurements, one should conclude these calculations up to reaching an interval $x_n = [a, b]$ of width 2^{-8} m (where $[a, b] = [\frac{s}{2^9}, \frac{s+2}{2^9}]$ for some $s \in \mathbb{Z}$). Then one checks whether $0 \in x_n$, and if so, Hawk-Eye should return the value LET, which in turn should lead to a replaying of the point in the tennis match. If $0 < a$ (or likewise $b < 0$), then Hawk-Eye can safely return the value IN (or likewise OUT), with the same consequences for play as are currently in use.

[9]There have been concerns about Hawk-Eye's accuracy in this respect, raised by Collins and Evans in [Col&Eva2008].

This illustrates our remark above, since adapting Hawk-Eye in this way corresponds to creating a morphism from $\mathbb{R}_{\#_\omega}$ to $(3_{\mathrm{nat}}, \mathcal{T}_{3_{\mathrm{nat}}})$. This type of problem occurs extremely frequently of course in applied mathematics, and one may think we are merely going over well-trodden grounds. But the topological cadre of reference is seldom explicitized, and systems like Hawk-Eye illustrate that awareness of this type of problem can still be improved.

As for visualizing our recommendation for Hawk-Eye, why not introduce a narrow gray line (representing LET) separating the outer part of the white line (representing IN) from the green (representing OUT). The gray line should be half on the former white band, and half on the former outer green expanse. Then the LET-situation occurs only if the ball is calculated to be largely in the outer green, but with a small overlap of gray and no overlap of white. This should be easily understandable to public and players.

A.2.1 Non-archimedean intermezzo: $\mathbb{C}_p$ For p a prime number, the complex p-adic numbers $\mathbb{C}_p$ can be defined constructively by first defining a valuation-norm on the algebraic p-adic numbers $\mathbb{A}_p = \{a_m \mid m \in \mathbb{N}\}$, and then defining $\mathbb{C}_p$ to be the metrical completion of $\mathbb{A}_p$ (see [MiRiRu1988]). $\mathbb{C}_p$ has an interesting feature which relates to our study of natural representations of metric spaces: it is a complete metric space which is not *spherically complete*. There is a series of shrinking closed spheres $(\overline{B}(c_m, q_m))_{m\in\mathbb{N}}$ where $c_m \in \mathbb{A}_p$, $q_m \in \mathbb{Q}$, $\overline{B}(c_{m+1}, q_{m+1}) \subseteq \overline{B}(c_m, q_m)$ for all m and yet $\bigcap_{m\in\mathbb{N}} \overline{B}(c_m, q_m) = \emptyset$.

This means, that if for $\mathbb{C}_p$ we proceed creating a natural space $(\mathcal{V}, \mathcal{T}_\#)$ as in remark 1.2.3, taking $V = \{\overline{B}(a_m, q) \mid q \in \mathbb{Q}, m \in \mathbb{N}\}$, then the ‘shriveling’ sequence $(\overline{B}(c_m, q_m))_{m\in\mathbb{N}}$ becomes a ‘new’ point in $\mathcal{V}$ which has no corresponding point in $\mathbb{C}_p$. In fact we do not know whether this $(\mathcal{V}, \mathcal{T}_\#)$ is metrizable. Classically, $(\mathcal{V}, \mathcal{T}_\#)$ can be weakly metrized as an extension of $\mathbb{C}_p$:

(in CLASS:) For $p, q \in \mathcal{V}$, put $d(p,q) = \lim_{n\to\infty} d(p_n, q_n)$, where for $n \in \mathbb{N}$ we let $d(p_n, q_n) \underset{D}{=} \inf(\{d(x,y) \mid x \in p_n, y \in q_n\})$. ($p_n, q_n$ are closed spheres in $(\mathbb{C}_p, d)$.).

In CLASS, the limit exists for elements of $\mathcal{V}$, and defines a metric which extends the non-archimedean metric d on $\mathbb{C}_p$.[10] We think $(\mathcal{V}, d)$ is spherically complete. However, $(\mathcal{V}, d)$ is not separable (so d does not metrize $(\mathcal{V}, \mathcal{T}_\#)$), and we do not know whether an alternative (constructive, separable) metric exists which might enable us to work with some form of spherical completion

[10] Consider that shriveling sphere-sequences p, q converge downwards to a limit radius r_1, r_2. If $r_1 < r_2$ then $d(p,q) = d(p_n, q_n) > r_2$ for some n. If $r_1 = r_2 > 0$ then $d(p,q) = 0$ or $d(p,q) = d(p_n, q_n) > r_2$ for some n. If $r_1 = r_2 = 0$ then $p, q \in \mathbb{C}_p$.

of $\mathbb{C}_p$ constructively as well. Our metrization theorem for star-finitary spaces (thm. 4.0.8) suggests that one should find out whether $(\mathcal{V}, \mathcal{T}_\#)$ is star-finitary. Whether all this leads to new topological spaces, we do not know. $\mathbb{C}_p$ with the usual metric topology is homeomorphic to Baire space, this might also hold for the $(\mathcal{V}, \mathcal{T}_\#)$ indicated here.[11]

A.2.2 The Cantor function and other morphisms to the binary reals (Continued from paragraph 1.3.2). To show the equivalence of the binary reals with the reals allowing a binary expansion, let $\{0,1\}^*$ be the set of finite sequences of elements of $\{0,1\}$. Now natural Cantor space is the natural subspace $(\mathcal{C}, \mathcal{T}_{\text{nat}})$ of Baire space $\mathcal{N}$ formed by the pre-natural space $(\{0,1\}^*, \#_\omega, \preceq_\omega)$ and its set of points $\mathcal{C}$ (see paragraph 2.0.3 for the precise definition). We also write σ_2 for the Cantor space.

Let us inductively define a surjective $\preceq$-morphism $f_{\text{evl},2}$ from $\mathcal{C}$ to $[0,1]_{\text{bin}}$. First we put $f_{\text{evl},2}(\bigcirc_\omega) = \bigcirc_{[0,1]} = [0,1] \in \mathrm{R}_{\mathbb{Q},\text{bin}}$. Now let $a = a_0, \ldots, a_{n-1} \in \{0,1\}$ where $f_{\text{evl},2}(a)$ has been defined and equals $[d,e] \in \mathrm{R}_{\mathbb{Q},\text{bin}}$. Then, with $a \star 0 = a_0, \ldots, a_{n-1}, 0$ and $a \star 1 = a_0, \ldots, a_{n-1}, 1$ we define $f_{\text{evl},2}(a \star 0) \underset{D}{=} [d, \frac{d+e}{2}]$ and $f_{\text{evl},2}(a \star 1) \underset{D}{=} [\frac{d+e}{2}, e]$. This inductively defines $f_{\text{evl},2}$ on all of $\{0,1\}^*$.

The reader can simply verify that $f_{\text{evl},2}$ is a surjective morphism from $\mathcal{C}$ to $[0,1]_{\text{bin}}$. Next, we pull back the apartness $\#_\mathbb{R}$ on $[0,1]_{\text{bin}}$ to $(\{0,1\}^*, \#_\omega, \preceq_\omega)$ by stipulating, for $a, b \in \{0,1\}^*$ that $a \#_\mathbb{R} b$ iff $f_{\text{evl},2}(a) \#_\mathbb{R} f_{\text{evl},2}(b)$. Then it is easy to see that $f_{\text{evl},2}$ is a $\preceq$-isomorphism from the natural space derived from $(\{0,1\}^*, \#_\mathbb{R}, \preceq_\omega)$ to $[0,1]_{\text{bin}}$.

Completely similar, we define $\sigma_3 = (\{0,1,2\}^*, \#_\omega, \preceq_\omega)$ and a surjective morphism $f_{\text{evl},3}$ from σ_3 to $[0,1]_{\text{ter}}$, such that pulling back $\#_\mathbb{R}$ using $f_{\text{evl},3}$ we see that $f_{\text{evl},3}$ is a $\preceq$-isomorphism from $\sigma_{3,\text{ter}} \underset{D}{=} (\{0,1,2\}^*, \#_\mathbb{R}, \preceq_\omega)$ to $[0,1]_{\text{ter}}$. We have use for an inverse of $f_{\text{evl},3}$ given explicitly, we therefore define the appropriate morphism $f^{-1}_{\text{evl},3}$ inductively. Put $f^{-1}_{\text{evl},3}([0,1]) = \bigcirc_\omega$. Next, suppose $f^{-1}_{\text{evl},3}(a)$ has been defined for a given interval $a = [\frac{n}{3^m}, \frac{n+1}{3^m}] \in \mathrm{R}_{\mathbb{Q},\text{ter}}$. Then we define: $f^{-1}_{\text{evl},3}([\frac{n+0}{3^{m+1}}, \frac{n+1}{3^{m+1}}]) = f^{-1}_{\text{evl},3}(a) \star 0$, $f^{-1}_{\text{evl},3}([\frac{n+1}{3^{m+1}}, \frac{n+2}{3^{m+1}}]) = f^{-1}_{\text{evl},3}(a) \star 1$ and $f^{-1}_{\text{evl},3}([\frac{n+2}{3^{m+1}}, \frac{n+3}{3^{m+1}}]) = f^{-1}_{\text{evl},3}(a) \star 2$.

The Cantor function f_{Can} is most easily defined as a morphism from σ_3 to σ_2. First take $a = a_0, \ldots, a_{n-1} \in \{0,2\}^*$. For all $i < n = \lg(a)$ let $b_i = \min(a_i, 1)$. Then put $f_{\text{Can}}(a) \underset{D}{=} b_0, \ldots, b_{n-1}$. Next let $b \in \{0,1,2\}^*, b \notin \{0,2\}^*$. Determine $a \in \{0,2\}^*$ and $c \in \{0,1,2\}^*$ such that $b = a \star 1 \star c$. Let $m = \lg(c)$, and let $\underline{\overline{0}}(m)$ be the sequence $0, \ldots, 0$ of length m. Put $f_{\text{Can}}(b) \underset{D}{=} f_{\text{Can}}(a) \star 1 \star \underline{\overline{0}}(m)$.

[11] Perhaps a nice subject for a Master's thesis?

Using $f^{-1}_{\text{evl},3}$ and $f_{\text{evl},2}$, this completely defines the Cantor function as a $\preceq$-morphism from $[0,1]_{\text{ter}}$ to $[0,1]_{\text{bin}}$. This morphism cannot however be extended to a $\preceq$-morphism from $[0,1]$ to $[0,1]_{\text{bin}}$, for the reasons described in chapter two. Proposition 2.3.2 shows how we can represent the Cantor function as a $\preceq$-morphism from $[0,1]$ to $[0,1]$.

We can however also define the Cantor function as a $\wr$-morphism from $[0,1]$ to $[0,1]_{\text{bin}}$. We leave this as a non-trivial exercise to the reader interested in applied mathematics and representation issues. For our narrative we turn to the promised property that any morphism f from $[0,1]$ to $[0,1]_{\text{bin}}$ is 'locally constant' around the f-original of a binary rational $\frac{n}{2^m}$ ($n \leq 2^m$).

PROPOSITION: Let f be a $\wr$-morphism from $[0,1]$ to $[0,1]_{\text{bin}}$ such that $x \leq_{\mathbb{R}} y$ implies $f(x) \leq_{\mathbb{R}} f(y)$, for $x, y \in [0,1]$, and such that $f(0) \equiv_{\mathbb{R}} 0$ and $f(1) \equiv_{\mathbb{R}} 1$. Suppose $z \in [0,1]$ is such that $f(z) \equiv_{\mathbb{R}} \frac{1}{2}$. Then there is a rational interval $[a,b] \in \mathrm{R}_{\mathbb{Q}}$ such that $f(x) \equiv_{\mathbb{R}} \frac{1}{2}$ for all $x \in [a,b]$.

PROOF: Clearly we can find a sequence $z' = ([a_n, b_n])_{n \in \mathbb{N}}$ of strictly shrinking rational intervals such that $z \equiv_{\mathbb{R}} z'$ ($a_n < a_{n+1} < b_{n+1} < b_n$ for all $n \in \mathbb{N}$). We determine a value of $f(z')$ which is not equal to the maximal dot $\bigcirc$, say $f(\overline{z'}(m)) \prec \bigcirc$. Then we have either `case 0`: $f(z')(m) \preceq_{\mathbb{R}} [0, \frac{1}{2}]$, then since $f(z') \equiv_{\mathbb{R}} \frac{1}{2}$ and f is $\leq_{\mathbb{R}}$-preserving, we see that $f(x) \equiv_{\mathbb{R}} \frac{1}{2}$ for all $x \in [b_m, b_{m-1}]$ or `case 1`: $f(z')(m) \preceq_{\mathbb{R}} [\frac{1}{2}, 1]$, then since $f(z') \equiv_{\mathbb{R}} \frac{1}{2}$ and f is $\leq_{\mathbb{R}}$-preserving, we see that $f(x) \equiv_{\mathbb{R}} \frac{1}{2}$ for all $x \in [a_{m-1}, a_m]$. (END OF PROOF)

COROLLARY: $[0,1]$ and $[0,1]_{\text{bin}}$ are not isomorphic.

We end with a number of statements which we do not prove (exercise):

(i) The n-ary reals are isomorphic to the m-ary reals, for $n, m \in \mathbb{N}$.

(ii) The n-ary reals can be identically embedded (meaning $f(x) \equiv_{\mathbb{R}} x$ for all x) in the m-ary reals iff there is a $b \geq 1$ in $\mathbb{N}$ such that m divides n^b.

(iii) Addition and multiplication cannot be represented as morphisms from $[0,1]_{\text{bin}}$ to itself, in other words the n-ary reals are not closed under addition and multiplication. This makes the n-ary reals unsuited for computational purposes, in our eyes.

(iv) Adding the extra digit -1 to the digits $0, \ldots, n-1$ solves the current problems with n-ary digital representation.

We believe the above to be of interest for representation issues of real numbers. One can also entertain an independent topological interest, see theorem 1.4.0 on pathwise$_{\text{nat}}$ connectedness and the next example of ContraCantor space. Finally, we note that the ternary reals play a very nice role in the proof of our metrization theorem for star-finitary spaces, see A.3.18.

A.2.3 ContraCantor space and $[0,1]$-embedded Cantor space

From [Kle&Ves1965] we can directly define a decidable countable subset (derived from what Andrej Bauer in [Bau2006] aptly calls the Kleene Tree) $K_{\text{bar}} = \{k_n \mid n \in \mathbb{N}\}$ of $\{0,1\}^*$ such that $k_n \preceq_\omega k_m$ implies $n = m$ for all $n, m \in \mathbb{N}$ and in addition such that in RUSS $\{[\![k]\!] \mid k \in K_{\text{bar}}\}$ is an open cover of $(\mathcal{C}, \mathcal{T}_{\text{nat}})$ which has no finite subcover.

We use K_{bar} to define ContraCantor space, which is a compact subspace $\mathfrak{I}_{[0,1]}$ of $[0,1]$ such that if we write $\mathcal{C}_{[0,1]}$ for the Cantor set (which is the standard embedding of Cantor space in $[0,1]$), we see: $d_{\mathbb{R}}(\mathfrak{I}_{[0,1]}, \mathcal{C}_{[0,1]}) = 0$ and yet in RUSS we also have $d_{\mathbb{R}}(x, \mathcal{C}_{[0,1]}) > 0$ for $\forall x \in \mathfrak{I}_{[0,1]}$.

First let us define $\mathcal{C}_{[0,1]}$. For this we first embed $\mathcal{C} = \sigma_2$ in σ_3, using the 'doubling' morphism $f_{\cdot 2}$ defined by: $f_{\cdot 2}(a_0, \ldots, a_{n-1}) = 2 \cdot a_0, \ldots, 2 \cdot a_{n-1}$, for $a_0, \ldots, a_{n-1}$ in $\{0,1\}^*$. Clearly $f_{\cdot 2}$ is an embedding morphism from σ_2 to σ_3. We put $\mathcal{C}_{\cdot 2} = f_{\cdot 2}(\mathcal{C})$. Next, we combine $f_{\cdot 2}$ with the morphism $f_{\text{evl},3}$ from σ_3 to $[0,1]_{\text{ter}}$, which we defined in the previous example.
We now put: $\mathcal{C}_{[0,1]} \underset{D}{=} f_{\text{evl},3} \circ f_{\cdot 2}(\mathcal{C})$.

Next we define the ContraCantor set $\mathfrak{I}_{[0,1]}$, by first considering the subspace $\sigma_{3,\text{contr}} = \{f_{\cdot 2}(k_n) \star 1 \star \alpha \mid k_n \in K_{\text{bar}}, \alpha \in \sigma_3\}$ of σ_3. Notice that $\sigma_{3,\text{contr}}$ lies apart from $f_{\cdot 2}(\mathcal{C})$, but at distance 0. In RUSS, $\sigma_{3,\text{contr}}$ is (the point set of) a fan. In INT and CLASS, $\sigma_{3,\text{contr}}$ is not closed, and to obtain a fan we must move to the closure of $\sigma_{3,\text{contr}}$.
We transfer this situation to $[0,1]$ by letting $\mathfrak{I}_{[0,1]}$ be the metrical closure of $f_{\text{evl},3}(\sigma_{3,\text{contr}})$.

PROPOSITION: $\mathfrak{I}_{[0,1]}$ is a BISH-compact subspace of $[0,1]$ with the property that $d_{\mathbb{R}}(\mathfrak{I}_{[0,1]}, \mathcal{C}_{[0,1]}) = 0$ and yet in RUSS also $d_{\mathbb{R}}(x, \mathcal{C}_{[0,1]}) > 0$ for all $x \in \mathfrak{I}_{[0,1]}$.

PROOF: By definition $\mathfrak{I}_{[0,1]}$ is complete and totally bounded, therefore BISH-compact, just like the Cantor set $\mathcal{C}_{[0,1]}$. Clearly $d_{\mathbb{R}}(\mathfrak{I}_{[0,1]}, \mathcal{C}_{[0,1]}) = 0$. In RUSS, we have $\mathfrak{I}_{[0,1]} = f_{\text{evl},3}(\sigma_{3,\text{contr}})$, since $\sigma_{3,\text{contr}}$ is already complete. This we see by considering in RUSS a convergent sequence $(x_n)_{n \in \mathbb{N}}$ in $\sigma_{3,\text{contr}}$. We construct a 'shadow' sequence $(y_n)_{n \in \mathbb{N}}$ in $\mathcal{C}_{\cdot 2} = f_{\cdot 2}(\mathcal{C})$ thus: for each $n \in \mathbb{N}$, x_n equals $f_{\cdot 2}(k_m) \star 1 \star \alpha$ for certain $m \in \mathbb{N}, \alpha \in \sigma_3$. We now put $y_n = f_{\cdot 2}(k_m) \star \underline{0}$ (where $\underline{0}$ is the infinite sequence $0, 0, \ldots \in \sigma_3$).

Clearly $(y_n)_{n\in\mathbb{N}}$ is convergent in $\mathcal{C}_{\cdot 2}$, we consider the limit $y\in\mathcal{C}_{\cdot 2}$. There is $z\in\mathcal{C}$ such that $y=f_{\cdot 2}(z)$, and since K_{bar} is a bar on $\mathcal{C}$, there is $n\in\mathbb{N}$ such that $z\preceq k_n$. This however implies that there is $N\in\mathbb{N}$ such that $x_m\preceq f_{\cdot 2}(k_n)\star\underline{1}$ for all $m\geq N$, showing that $(x_n)_{n\in\mathbb{N}}$ converges to a limit in $\sigma_{3,\mathrm{contr}}$.

Therefore we obtain in RUSS: $d_{\mathbb{R}}(x,\mathcal{C}_{[0,1]})>0$ for all $x\in\mathfrak{I}_{[0,1]}$. (END OF PROOF)

This situation in RUSS, where two compact spaces have distance 0 and yet are apart, is well known. Perhaps less known is how it bears on our discussion of inductive morphisms and the pointwise problems arising in BISH even if we inductivize our definitions, see paragraph 3.4.0.

A.2.4 The spraid of uniformly continuous real-valued functions on $[0,1]$

Brouwer already showed how to build the space $C^{\mathrm{unif}}([0,1],\mathbb{R})_{\mathrm{nat}}$ of uniformly continuous real-valued functions on $[0,1]$ as a spread (see [Bro1975], [vDal2003]). The basic idea is to consider the graph of such functions, one then sees that this graph is a compact subspace of the real plane which is homeomorphic to the line segment $[0,1]$. For a uniformly continuous real-valued f on $[0,1]$, the graph G_f can thus be built as a line segment in the x,y-plane twisting from the vertical line $x=0$ to the vertical line $x=1$ without 'doubling back' in the horizontal sense. We can approximate the graph G_f with step-by-step growing precision, by forming at stage n a 'tape' T^f_n of rectangles, each with height 2^{-n}, which encloses G_f and which runs from the line $x=0$ to the line $x=1$. The rectangles all have equal width 2^{-m} where m is determined by the uniform-continuity modulus of f. And we specify that the corner-coordinates of these rectangles are taken from the set $\{(a,b)\,|\,[2^{n+1}\cdot a\in\mathbb{N}\wedge 2^{m+1}\cdot b\in\mathbb{N}]\}$. We can do this in such a way that each rectangle of T^f_{n+1} lies completely within a rectangle of T^f_n, for each $n\in\mathbb{N}$.

We turn to the whole space $C^{\mathrm{unif}}([0,1],\mathbb{R})_{\mathrm{nat}}$, which we wish to represent as a spraid. Abstracting from the specified f above, the properties of these 'tapes' $(T^f_n)_{n\in\mathbb{N}}$ can now be formulated in such a way that we can take the countable collection of all such tapes as the set of basic dots of our desired spraid. Then the definition of the refinement relation and the apartness relation is a direct consequence of our plan thus far. If we stick to this plan, the resulting points will be seen to represent elements of $C^{\mathrm{unif}}([0,1],\mathbb{R})_{\mathrm{nat}}$, and vice versa, each element of $C^{\mathrm{unif}}([0,1],\mathbb{R})_{\mathrm{nat}}$ will correspond to a point in this spraid.

This indicates how to build $C^{\mathrm{unif}}([0,1],\mathbb{R})_{\mathrm{nat}}$ as a spraid. We do not go into this further here, but leave this as a challenge to the reader.

A.2.5 A counterexample in CLASS illustrating $\mathbf{AC}_{11}$ We elaborate on our remarks in 1.2.2, by giving an example in CLASS of a natural space $(\mathcal{V}, \mathcal{T}_{\#})$ where for all $x \in \mathcal{V}$ there is a $y \equiv x$ in $\mathcal{V}$ such that $\llbracket y_n \rrbracket$ is open for all $n \in \mathbb{N}$, and yet $(\mathcal{V}, \mathcal{T}_{\#})$ is not a basic neighborhood space.
For this, we use the 'open' refinement relation $\preceq^{\circ}_{\mathbb{R}}$, and we turn again to $[0,1]$ and $[0,1]_{\text{bin}}$ (see defs. 1.0.8 and 1.3.1). For each $a \in [0,1]_{\mathrm{R}}$ we introduce a copy a^* which we call *starred* and we put:

$$V \underset{D}{=} \{\, [\tfrac{1}{2} - 2^{-n-2}, \tfrac{1}{2} + 2^{-n-2}] \mid n \in \mathbb{N}\} \cup [0,1]_{\mathrm{R,bin}} \cup \{ [p,q]^* \mid [p,q] \in [0,1]_{\mathrm{R}} \mid p \geq \tfrac{1}{2} \vee q \leq \tfrac{1}{2}\}.$$

For starred $a^* \in V$, where $a \in [0,1]_{\mathrm{R}}$, put $i(a^*) = a$. And for unstarred $c \in V$ put $i(c) = c$. For $c \in V$ put $c \preceq [0,1] = \bigcirc_V$. Then for $c, d \in V, d \neq [0,1]$ we put $c \preceq d$ iff (c is starred $\longleftrightarrow$ d is starred) $\wedge$ $i(c) \preceq^{\circ}_{\mathbb{R}} i(d)$. Finally put $c \# d$ iff $i(c) \#_{\mathbb{R}} i(d)$. Now let $(\mathcal{V}, \mathcal{T}_{\#})$ be the natural space derived from $(V, \#, \preceq)$. We identify the elements of $\mathcal{V}$ with the real numbers that they obviously represent.

claim (in CLASS) For all $x \in \mathcal{V}$ there is $\mathcal{V} \ni y \equiv x$ where $\llbracket y_n \rrbracket$ is open for all $n \in \mathbb{N}$.

proof If $x \# \frac{1}{2}$, then we can trivially find $y \equiv x$ in $\mathcal{V}$ such that y_n is starred and moreover $\llbracket y_n \rrbracket$ is open for all $n \in \mathbb{N}$ (by our remarks on $\preceq^{\circ}_{\mathbb{R}}$, in 1.0.8). If $x \equiv \frac{1}{2}$, then take y given by $y_n = [\frac{1}{2} - 2^{-n-2}, \frac{1}{2} + 2^{-n-2}]$ for $n \in \mathbb{N}$, and so we are done. (end of claim-proof)

claim (in BISH) $(\mathcal{V}, \mathcal{T}_{\#})$ is not isomorphic to a basic-open space.

proof Let $(\mathcal{W}, \mathcal{T}_{\#_2})$ be a basic-open space such that $(\mathcal{V}, \mathcal{T}_{\#})$ is isomorphic to $(\mathcal{W}, \mathcal{T}_{\#_2})$ under isomorphism f with inverse g. Put $h = g \circ f$, then h is an identical automorphism on $(\mathcal{V}, \mathcal{T}_{\#})$ where for all $x \in \mathcal{V}$ in addition $h(x)_n = h(\overline{x}(n))$ is a basic neighborhood of $h(x)$ for all $n \in \mathbb{N}$ (w.l.o.g. h is a $\wr$-morphism). Consider the point $x \equiv \frac{1}{2}$ given by $x_n = [\frac{1}{2} - 2^{-n-2}, \frac{1}{2}]^*$ for $n \in \mathbb{N}$. By the above and since $h(x) \equiv \frac{1}{2}$ is a point, there must be $n \in \mathbb{N}$ such that $h(x)_n$ is of the form $[\frac{1}{2} - 2^{-m-2}, \frac{1}{2} + 2^{-m-2}]$ for some $m \in \mathbb{N}$. Now look at the point $y \equiv \frac{1}{2} - 2^{-n-2}$ in $\mathcal{V}$ given by $\overline{y}(n) = \overline{x}(n)$ and $y_{n+s} = [\frac{1}{2} - 2^{-n-2}, \frac{1}{2} - 2^{-n-2} + 2^{-n-2-s}]^*$, for all $s \in \mathbb{N}$. Since $\overline{y}(n) = \overline{x}(n)$, we have $h(y)_n = [\frac{1}{2} - 2^{-m-2}, \frac{1}{2} + 2^{-m-2}]$. Now the only way to $\preceq$-refine $h(y)_n$ to a point equivalent to y is by using basic dots in $[0,1]_{\mathrm{R,bin}}$. But these dots do not form a neighborhood of y, contradiction. (end of claim-proof)

Thus our example $(\mathcal{V}, \mathcal{T}_{\#})$ is not a basic neighborhood space, and yet in CLASS: for all $x \in \mathcal{V}$ there is a $y \equiv x$ in $\mathcal{V}$ such that $\llbracket y_n \rrbracket$ is open for all $n \in \mathbb{N}$.

REMARK: The ‘reason’ that $(\mathcal{V}, \mathcal{T}_{\#})$ is not a basic neighborhood space lies in the fact that in the statement: ‘$\forall x \in \mathcal{V}\, \exists y \equiv x\, \forall n \in \mathbb{N}\, [\llbracket y_n \rrbracket \in \mathcal{T}_{\#}]$’ the information is not given ‘continuously’, that is by a morphism. In fact one can question the statement precisely on this account, since by taking a z resembling the x in the claim-proof above, we see that we have no method to assign to z an appropriate y unless we already know in advance the infinite behaviour of z (of which in general we are ignorant).

This is precisely the gist of the intuitionistic axiom $\mathbf{AC}_{11}$ which states that if we really know $\forall x\, \exists y\, [P(x, y)]$, then this information must be given continuously (by a morphism), otherwise we will always have sequences x for which we have no method to produce y with $P(x, y)$. (END OF REMARK)

A.3 PROOFS AND ADDITIONAL DEFINITIONS

A.3.0 Proof of theorem 1.0.8

THEOREM: (repeated from 1.0.8) $(\mathbb{R}_{\mathrm{nat}}, \mathcal{T}_{\#_{\mathbb{R}}})$ is a natural space which is homeomorphic to the topological space of the real numbers $\mathbb{R}$ equipped with the usual metric topology.

PROOF: We are a bit free here, since for a classical theorist we should first move to the quotient space of equivalence classes. We consider this a cumbersome practice, and prefer to give a 'direct' proof. In this proof we consider $\mathbb{R}$ to be given as the collection of all Cauchy-sequences in $(\mathbb{Q}, d_{\mathbb{R}})$.

To any $x = ([a_n, b_n])_{n\in\mathbb{N}} \in \mathbb{R}_{\mathrm{nat}}$ we assign the Cauchy-sequence $f(x) = (a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$. We leave it to the reader to verify that (i) for any $y \in \mathbb{R}$ there is a $z \in \mathbb{R}_{\mathrm{nat}}$ with $f(z) = y$, (ii) for all $x, y \in \mathbb{R}_{\mathrm{nat}}$ we have $x \#_{\mathbb{R}} y$ iff $d(f(x), f(y)) > 0$. Therefore f is surjective and injective.

It is easy to see that f is continuous, so to see that f is a homeomorphism we must show that f is open. For this let $U \subseteq \mathbb{R}_{\mathrm{nat}}$ be $\#$-open. We must show that $f(U)$ is open in $(\mathbb{R}, d_{\mathbb{R}})$. For this let $z \in f(U)$, determine $x = ([a_n, b_n])_{n\in\mathbb{N}} \in U$ such that $f(x) = z$. It is not hard to construct a $y = ([c_n, d_n])_{n\in\mathbb{N}} \in \mathbb{R}_{\mathrm{nat}}$ such that $y \equiv_{\mathbb{R}} x$ and for all $n \in \mathbb{N}$: $c_{n+1} - c_n > \frac{1}{4} \cdot (d_n - c_n)$ and $d_n - d_{n+1} > \frac{1}{4} \cdot (d_n - c_n)$.

Since U is open and $y \equiv_{\mathbb{R}} x \in U$, we can find $n \in \mathbb{N}$ such that $[\![[c_n, d_n]]\!] \subseteq U$. So $f(U)$ contains the interval $[c_n, d_n]$, and also $z = f(x) \equiv f(y) \in [c_{n+1}, d_{n+1}]$. From this we conclude that $f(U)$ contains the metric ball $B(z, \frac{1}{4} \cdot (d_n - c_n))$, showing that $f(U)$ is open. (END OF PROOF)

A.3.1 Proof of theorem 1.2.2

For this proof we use theory from later sections. Especially simplifying is theorem 2.2.0 which states that every natural space is spreadlike, and even isomorphic to a spread whose tree is $(\mathbb{N}^*, \preceq_\omega)$. The corollary from its proof in A.3.4 shows that a basic-open space is isomorphic to a basic-open spread whose tree is $(\mathbb{N}^*, \preceq_\omega)$. We also use the terminology of later chapters, notably chapter three.

DEFINITION: Let $(\mathcal{V}, \mathcal{T}_{\#})$ be a spread derived from $(V, \#, \preceq)$, and let B be a bar on V (see def. 3.1.0). Then we say that B is a *thin bar* on V iff for all $a \in B$ and

$b \prec a$ we have that $b \notin B$. (Then for a successor point $x \in \mathcal{V}$ there is a unique $n \in \mathbb{N}$ with $x_n \in B$, see A.3.17 (**a**)). (END OF DEFINITION)

THEOREM: (in CLASS, INT and RUSS, repeated from 1.2.2):
Let f be a continuous function from a natural space $(\mathcal{V}, \mathcal{T}_{\#_1})$ to a basic neighborhood space $(\mathcal{W}, \mathcal{T}_{\#_2})$. Then there is a natural morphism g from $(\mathcal{V}, \mathcal{T}_{\#_1})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$ such that for all x in $\mathcal{V}$: $f(x) \equiv_2 g(x)$.

PROOF: We give a unified proof for CLASS, INT and RUSS derived from the common Lindelöf axiom $\mathbf{BDD}^*$ defined in A.4.12, which states that every bar on $\mathcal{N}$ (or equivalently $\mathbb{N}^*$) descends from a thin decidable bar.

By theorem 2.2.0 and its corollary (proved in A.3.4) it suffices to prove the theorem for the case where $(\mathcal{V}, \mathcal{T}_{\#_1})$ is a spread derived from $(\mathbb{N}^*, \#_1, \preceq_\omega)$ and $(\mathcal{W}, \mathcal{T}_{\#_2})$ is a basic-open spread derived from $(\mathbb{N}^*, \#_2, \preceq_\omega)$. So in the following keep in mind that $[\![b]\!]$ is $\#_2$-open for every $b \in W = \mathbb{N}^*$.

We will inductively define a sequence of thin bars $(C_n)_{n \in \mathbb{N}}$ on $V = \mathbb{N}^*$ and simultaneously construct the desired morphism g, as follows. First let $n = 0$, put $C_0 = \{\bigcirc_V\}$ and put $g(\bigcirc_V) = \bigcirc_W$. Let $c \in C_0$, then since f is a continuous function, for all $x \in [c] = \mathcal{V}$ there are $s \in \mathbb{N}$ with $x_s \prec c$ and $b \in \propto(g(c)) = \propto(\bigcirc_W)$ such that $f([x_s]) \subseteq [\![b]\!]$.

Therefore the set $B_1 = \{a \in \mathbb{N}^* \mid \exists c \in C_0 [a \prec c \wedge \exists b \in \propto(g(c))[f([a]) \subseteq [\![b]\!]]\}$ is a bar on $V = \mathbb{N}^*$. By $\mathbf{BDD}^*$ we find a decidable thin bar C_1 on $V = \mathbb{N}^*$ from which B_1 descends. Then for all $a \in C_1$ we have: $\exists b \in \propto(g(\bigcirc_V))[f([a]) \subseteq [\![b]\!]]$.

Using countable choice ($\mathbf{AC}_{00}$) we can now assign to each $a \in C_1$ a value $g(a)$ in $\propto(g(\bigcirc_V)) = \propto(\bigcirc_W)$ such that $f([a]) \subseteq [\![g(a)]\!]$. Also, to each $b \in V$ for which there is $d \prec b \prec c$ with $d \in C_1, c \in C_0$ we assign: $g(b) = g(c) = \bigcirc_W$.

We can repeat this process for $n = 1$ and C_1. For let $c \in C_1$, then since f is continuous, for all $x \in [c]$ there are $s \in \mathbb{N}$ with $x_s \prec c$ and $b \in \propto(g(c))$ such that $f([x_s]) \subseteq [\![b]\!]$.

Therefore the set $B_2 = \{a \in \mathbb{N}^* \mid \exists c \in C_1 [a \prec c \wedge \exists b \in \propto(g(c))[f([a]) \subseteq [\![b]\!]]\}$ is a bar on $V = \mathbb{N}^*$. By $\mathbf{BDD}^*$ we find a decidable thin bar C_2 on V from which B_2 descends. Then for all $a \in C_2$ we have: $\exists c \in C_1 \exists b \in \propto(g(c))[f([a]) \subseteq [\![b]\!]]$

Using countable choice ($\mathbf{AC}_{00}$) we can now assign to each $a \in C_2$ a value $g(a)$ in $\propto(g(c))$ (where $c \in C_1$ is such that $a \prec c$) such that $f([a] \subseteq [\![g(a)]\!]$. Also, to each $b \in V$ for which there is $d \prec b \prec c'$ with $d \in C_2, c' \in C_1$ we assign: $g(b) = g(c')$.

We can repeat this process for $n=2$ and C_2, etc. In this way, by using dependent countable choice ($\mathbf{DC}_1$) we can construct $(C_n)_{n\in\mathbb{N}}$ and simultaneously define g on all of $V=\mathbb{N}^*$. It is easy to see that $g(x)\equiv_2 f(x)$ for all $x\in\mathcal{V}$, which by continuity of f shows that g is a morphism. (END OF PROOF)

A.3.2 Proof of theorem 1.2.3

THEOREM: (repeated from 1.2.3) Every complete separable metric space (X,d) is homeomorphic to a basic-open space $(\mathcal{V},\mathcal{T}_{\#})$.

PROOF: The rough idea is simple: for a separable metric space (X,d) with dense subset $(a_n)_{n\in\mathbb{N}}$, let for each $n,s\in\mathbb{N}$ a basic dot be the open sphere $B(a_n,2^{-s}) = \{x\in X \mid d(x,a_n)<2^{-s}\}$. Then we can take as set of dots $V = \{B(a_n,2^{-s}) \mid n,s\in\mathbb{N}\}\cup\{\bigcirc_V\}$. The technical trouble now is to define $\#$ and $\preceq$ constructively, since in general even for $s>t$ the containment relation $B(a_n,2^{-s})\subseteq B(a_m,2^{-t})$ is not decidable. However, this containment relation has an enumerable subrelation which also does the trick. This because for all (a_n,s) and (a_m,t) with $s>t$ there is $i\in\{0,1\}$ such that:

$$(i=0\wedge d(a_n,a_m)<2^{-t}-2^{-s}) \text{ or } (i=1\wedge d(a_n,a_m)>2^{-t}-2^{-s}-2^{-2s})$$

Using $\mathbf{AC}_{00}$ (countable choice) we can define a function h fulfilling the above statement. Now we put $B(a_n,2^{-s})\prec B(a_m,2^{-t})$ iff $h((a_n,s),(a_m,t))=0$. Likewise we define $\#$, since for all (a_n,s) and (a_m,t) there is $j\in\{0,1\}$ such that:

$$(j=0\wedge d(a_n,a_m)<2^{-s}+2^{-t}+2^{-s-t}) \text{ or } (j=1\wedge d(a_n,a_m)>2^{-s}+2^{-t}+2^{-s-t-1})$$

Using $\mathbf{AC}_{00}$ we can define a function g fulfilling the above statement. Now we simply put $B(a_n,2^{-s})\#B(a_m,2^{-t})$ iff $g((a_n,s),(a_m,t))=1$.
It is not difficult to see that $(V,\#,\preceq)$ generates a basic-open natural space which is homeomorphic to (X,d). (END OF PROOF)

REMARK: Notice that by our definition of $\#$, if $B(a_n,2^{-s})\#B(a_m,2^{-t})$, then $d(a_n,a_m)>2^{-s}+2^{-t}+2^{-s-t-1}$). This is an important detail for proving theorem 3.4.3. (END OF REMARK)

COROLLARY: In CLASS, INT and RUSS the following holds:

(i) A continuous function f from a natural space $(\mathcal{W},\mathcal{T}_{\#})$ to a complete metric space (X,d) can be represented by a morphism from $(\mathcal{W},\mathcal{T}_{\#})$ to a basic neighborhood space $(\mathcal{V},\mathcal{T}_{\#})$ homeomorphic to (X,d), by theorem 1.2.2.

(ii) A representation of a complete metric space as a basic neighborhood space is unique up to isomorphism.

In BISH the following holds:

(iii) If (X, d) and $(\mathcal{V}, \mathcal{T}_{\#})$ are as above in the theorem, then we can define a metric d' on $(\mathcal{V}, \mathcal{T}_{\#})$ (see def. 4.0.0) by defining $d'(x, y) = d(h(x), h(y))$ for $x, y \in \mathcal{V}$ and h a homeomorphism from $(\mathcal{V}, \mathcal{T}_{\#})$ to (X, d). This metric can be obtained as a morphism from $(\mathcal{V} \times \mathcal{V}, \mathcal{T}_{\#})$ to $\mathbb{R}_{\text{nat}}$ by the construction of $(\mathcal{V}, \mathcal{T}_{\#})$. We then see that the apartness topology and the metric d'-topology coincide, in other words $(\mathcal{V}, \mathcal{T}_{\#})$ is metrizable. We conclude: on (this natural representation of) a complete metric space, the metric topology coincides with the apartness topology.

A.3.3 Proof of theorem 1.4.0

THEOREM: (repeated from 1.4.0) $\mathbb{R}_{\text{bin}}$ (as well as $\mathbb{R}_{\text{ter}}$, $\mathbb{R}_{\text{dec}}$) is a pathwise$_{\text{nat}}$ connected space which is not arcwise$_{\text{nat}}$ connected.

PROOF: A detailed constructive proof for $\mathbb{R}_{\text{ter}}$ is given in [Waa1996], this construction can be literally transposed to our setting to show that $\mathbb{R}_{\text{ter}}$ is pathwise$_{\text{nat}}$ connected. That $\mathbb{R}_{\text{ter}}$ is not arcwise$_{\text{nat}}$ connected follows from our work in example A.2.2. We sketch an alternative proof using the Cantor function (see example A.2.2 and paragraph 1.3.2).

Suppose $x <_{\mathbb{R}} y \in [0,1]_{\text{bin}}$, we want to show that there is a morphism f from $[0,1]$ to $[0,1]_{\text{bin}}$ such that $f(0) \equiv_{\mathbb{R}} x$ and $f(1) \equiv_{\mathbb{R}} y$. It is not so difficult to see that $[0,1]_{\text{bin}}$ is isomorphic to $\{z \in [0,1]_{\text{bin}} \mid x \leq_{\mathbb{R}} z \leq_{\mathbb{R}} y\}$ under an isomorphism g with $g(0) \equiv x$ and $g(1) \equiv y$. This means that we can take $f \underset{D}{=} g \circ f_{\text{Can}}$. For $y <_{\mathbb{R}} x$ we can mirror this argument.

Now if $x, y \in [0,1]_{\text{bin}}$ such that at stage n we still cannot determine $x \#_{\mathbb{R}} y$, then we can still start constructing f, sending initial values of $0 \in [0,1]$ to initial values of x and intial values of $1 \in [0,1]$ to initial values of y, in such a way that if at any later stage m we see $x(m) \#_{\mathbb{R}} y(m)$, then we can continue as above in the case where $x <_{\mathbb{R}} y$ or $y <_{\mathbb{R}} x$.

Of course, to complete the proof one must show that f_{Can} can indeed be given as a $\wr$-morphism from $[0,1]$ to $[0,1]_{\text{bin}}$ (we left this as a non-trivial exercise in example A.2.2). One also needs to extend the proof for $[0,1]_{\text{bin}}$ to all of $\mathbb{R}_{\text{bin}}$, which involves some extra work since $\mathbb{R}_{\text{bin}}$ is not closed under addition and multiplication. Finally, one can use the exercise that all the n-ary reals are isomorphic to transfer the pathwise$_{\mathrm{nat}}$ connectedness of $\mathbb{R}_{\mathrm{bin}}$ to all the n-ary reals. (END OF PROOF)

A.3.4 Proof of theorem 2.2.0

We need a preparatory definition.

DEFINITION: Let $(\mathcal{V}, \mathcal{T}_{\#})$ be a natural space derived from $(V, \#, \preceq)$. We introduce a formal element $\otimes$ not contained in V, and put $\otimes \# a$ for all $\bigcirc_V \neq a \in V$, and $\neg(\otimes \# \otimes)$. Also put $V_{\otimes} = V \cup \{\otimes\}$. Then the set $\#_{V_{\otimes}} = \{(c,d) \in V_{\otimes} \times V_{\otimes} \mid c \# d\}$ is countable.[12]

We say that $e : \mathbb{N} \to \#_{V_{\otimes}}$ is a *pregrade* on V iff e is an enumeration of either $\#_{V_{\otimes}}$ or of $\#_V = \{(c,d) \in V \times V \mid c \# d\}$ [13]. Let e be a pregrade on V, then for $n \in \mathbb{N}$ and a basic dot a we say that a chooses on $e_n = (c,d)$ iff $a \# c$ and/or $a \# d$. We now define a decidable gradation on basic dots as follows: for every $n \in \mathbb{N}$ a basic dot a is of *e-grade n* (notation $\mathrm{gd}_e(a) \geq n$) iff a chooses on e_i for every $i < n$.

In addition, let $v : \mathbb{N} \to V$ be an injective enumeration of V. For each $n \in \mathbb{N}$ we define the decidable set $B_n^{v,e} = \{v_m \in V \mid m \geq n \wedge \mathrm{gd}_e(v_m) \geq n\}$.
Finally, for $n \in \mathbb{N}$ a basic dot a is of (v,e)-*grade n* iff a is in $B_n^{v,e}$ and not in $B_{n+1}^{v,e}$. This is also decidable, so every basic dot has a unique decidable (v,e)-grade. Notice that $B_0^{v,e} = V$, and $\bigcirc_V \notin B_1^{v,e}$. (END OF DEFINITION)

LEMMA: Let $(\mathcal{V}, \mathcal{T}_{\#})$ be a natural space derived from $(V, \#, \preceq)$. Let $v : \mathbb{N} \to V$ be an injective enumeration of V, and let $e : \mathbb{N} \to \#_{V_{\otimes}}$ be a pregrade on V. Then a sequence $x = x_0 \succeq x_1 \succeq x_2 \ldots$ of basic dots in V forms a point in $\mathcal{V}$ iff for every $n \in \mathbb{N}$, there is an $m \in \mathbb{N}$ such that $x_m \in B_n^{v,e}$.

PROOF: The proof is a simple checking of the definitions, which we leave to the reader as an exercise. (END OF PROOF)

We are now ready to prove the main theorem:

THEOREM: (repeated from 2.2.0) Every natural space is spreadlike. In fact, every natural space $(\mathcal{V}, \mathcal{T}_{\#})$ is isomorphic to a spread $(\mathcal{W}, \mathcal{T}_{\#})$ whose tree is $(\mathbb{N}^*, \preceq_\omega)$.

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[12] This is the reason for introducing $\otimes$, since for a space containing just one point up to equivalence, $\#_V$ is empty, and there are many spaces of which we don't know whether they contain more than one point.

[13] This last addition is to avoid cumbersome notation if we have a space containing at least two points.

PROOF: Let $(\mathcal{V}, \mathcal{T}_\#)$ be a natural space derived from $(V, \#, \preceq)$. Let $v : \mathbb{N} \to V$ be an injective enumeration of V, and let $e : \mathbb{N} \to \#_{V_\otimes}$ be a pregrade on V. For $d \in V$ we put $\{d\}_\preceq = \{c \in V \mid c \preceq d\}$.

We inductively define a sequence of functions $(h_n)_{n\in\mathbb{N}}$ from ${}^n\mathbb{N}^* = \{a \in \mathbb{N}^* \mid \lg(a) = n\}$ to $B_n^{v,e}$ such that for b in ${}^{n+1}\mathbb{N}^*$ and a in ${}^n\mathbb{N}^*$: if $b \in \propto(a)$ then $h_{n+1}(b) \prec h_n(a)$. In addition, the functions $(h_n)_{n\in\mathbb{N}}$ will be 'surjective enough', meaning that every point in $\mathcal{V}$ will be represented in the end, when we join all the h_n's to a single morphism h from Baire space to $(\mathcal{V}, \mathcal{T}_\#)$.

First, let h_0 be the function from $\{\bigcirc_\omega\}$ to $B_0^{v,e}$ given by $h_0(\bigcirc_\omega) = \bigcirc_V$.

We turn to $n=1$. We can determine a bijection $g_{\bigcirc_\omega} = h_1$ from $\propto(\bigcirc_\omega) = {}^1\mathbb{N}^*$ to $B_1^{v,e}$. Trivially for b in ${}^1\mathbb{N}^*$ and a in ${}^0\mathbb{N}^* = \{\bigcirc_\omega\}$ we have: if $b \in \propto(a)$ then $h_1(b) \prec h_0(a)$.

Next we turn to $n=2$. For $a \in {}^1\mathbb{N}^*$ we look at $h_1(a) \in V$. Remember that $\{h_1(a)\}_\prec = \{b \in V \mid b \prec h_1(a)\}$. We hold: $B_a \underset{D}{=} B_2^{v,e} \cap \{h_1(a)\}_\prec$ is a countable set (since it is infinite, and all the relevant relations are decidable). Therefore we can determine a bijection g_a from $\propto(a)$ to B_a. This means that we can take $h_2 = \bigcup_{a\in{}^1\mathbb{N}^*} g_a$, and see that for b in ${}^2\mathbb{N}^*$ and a in ${}^1\mathbb{N}^*$ we have: if $b \in \propto(a)$ then $h_2(b) \prec h_1(a)$.

Now we are in business, since for $n=3$ and so on the above process can be continued verbatim, changing only the index n. This yields two sequences of functions, which arise intertwinedly. The first sequence is $(h_n)_{n\in\mathbb{N}}$ from ${}^n\mathbb{N}^* = \{a \in \mathbb{N}^* \mid \lg(a) = n\}$ to $B_n^{v,e}$ such that for b in ${}^{n+1}\mathbb{N}^*$ and a in ${}^n\mathbb{N}^*$: if $b \in \propto(a)$ then $h_{n+1}(b) \prec h_n(a)$. The second sequence is $(g_a)_{a\in\mathbb{N}^*}$, where for $a \in {}^n\mathbb{N}^*$, g_a is a bijection from $\propto(a)$ to $B_a = B_{n+1}^{v,e} \cap \{h_n(a)\}_\prec$.
(We do not really use $\mathbf{DC}_1$ since all these functions can be found canonically once we have fixed our enumerations v and e).

claim If for $a \in {}^n\mathbb{N}^*$ we put $h(a) = h_n(a)$, then h is a surjective morphism from Baire space $\mathcal{N}$ to $(\mathcal{V}, \mathcal{T}_\#)$. If we introduce an apartness $\#_W$ on $\mathbb{N}^*$ by putting $a \#_W b$ iff $h(a) \#_V h(b)$, then h is an injective morphism from the spread $(\mathcal{W}, \mathcal{T}_\#)$ derived from $(\mathbb{N}^*, \#_W, \preceq_\omega)$ to $(\mathcal{V}, \mathcal{T}_\#)$.

proof By the above lemma, h is surjective since g_a is a bijection from $\propto(a)$ to B_a for $a \in \mathbb{N}^*$. Now $(\mathcal{W}, \mathcal{T}_\#)$ is a spread, since for $a \in {}^n\mathbb{N}^*$ we have $h(a) \in B_n^{v,e}$, so infinite $\prec_\omega$-trails define points. (end of claim-proof)

To complete the proof, we need to show that h is an isomorphism from $(\mathcal{W}, \mathcal{T}_\#)$ to $(\mathcal{V}, \mathcal{T}_\#)$. Therefore we need to construct an inverse h^{inv} for h.

In general this inverse h^{inv} can only be constructed as a trail morphism (see def. 1.1.4), in other words a refinement morphism on the trail space of $(\mathcal{V}, \mathcal{T}_{\#})$. So we turn to the trail space $(\mathcal{V}^{\wr}, \mathcal{T}_{\#}^{\wr})$, derived from $(V^{\wr}, \#^*, \preceq^*)$. We also use the above definition and lemma, and the sequences of functions $(h_n)_{n\in\mathbb{N}}$ and $(g_a)_{\bigcirc\neq a\in\mathbb{N}^*}$ above.

To start put $h^{\mathrm{inv}}(\bigcirc^*)=\bigcirc_{\omega}$. Next, let $a=a_0,\ldots,a_n$ be in $V^{\wr}$ (so $a_0\succ\ldots\succ a_n$). We associate to a a unique 'minimal grade sequence' $p_a=p_0,\ldots,p_{j-1}$ with $j\leq n+1$, where p_a is a subsequence of a. First we take $i_0=\mu k\leq n[a_k\in B_1^{V,e}]$ if such k exists. If such k does not exist, we are done and the minimal grade sequence associated to a is the empty sequence and $j=0$. Next, if $j\neq 0$ we take $i_1=\mu k\leq n[k>i_0\wedge a_k\in B_2^{V,e}]$ if such k exists (else $j=1$ and we are done). And generally let $i_s=\mu k\leq n[k>i_{s-1}\wedge a_k\in B_{s+1}^{V,e}]$, until we have exhausted a. Then $p_a=p_0,\ldots,p_{j-1}=a_{i_0},\ldots,a_{i_{j-1}}$.

To define $h^{\mathrm{inv}}(a)$ we use the minimal grade sequence $p_0,\ldots,p_{j-1}$. If $j=0$ then we put $h^{\mathrm{inv}}(a)=\bigcirc_{\omega}$. Else, we can first turn to p_0. Since h_1 is a bijection, we can determine $b_0=h_1^{-1}(p_0)\in{}^1\mathbb{N}^*$. If $j=1$ we are done, else we know that g_{b_0} is a bijection from $\propto(b_0)$ to $B_{p_0}=B_2^{V,e}\cap\{p_0\}_{\prec}$. Then we can put $b_1=g_{b_0}^{-1}(p_1)$. In this way we can continue, putting $b_{s+1}=g_{b_s}^{-1}(p_{s+1})$ for all $s<j-1$. Finally we put $h^{\mathrm{inv}}(a)=b_0,\ldots,b_{j-1}$. Since a is arbitrary, this suffices to define h^{inv} on $V^{\wr}$, and also on $(\mathcal{V}, \mathcal{T}_{\#})$ by putting $h^{\mathrm{inv}}(x)=h^{\mathrm{inv}}(\overline{\overline{x}}(0)), h^{\mathrm{inv}}(\overline{\overline{x}}(1)),\ldots$ for $x\in\mathcal{V}$.

claim h^{inv} is a trail morphism from $(\mathcal{V}, \mathcal{T}_{\#})$ to $(\mathcal{W}, \mathcal{T}_{\#})$ such that $h^{\mathrm{inv}}\circ h(x)\equiv x$ for all $x\in\mathcal{W}$ and $h\circ h^{\mathrm{inv}}(y)\equiv y$ for all $y\in\mathcal{V}$.

proof The only real concern for showing that h^{inv} is actually a $\wr$-morphism with the correct properties, is that h^{inv} sends points in $\mathcal{V}$ to points in $\mathcal{W}$. So let $x\in\mathcal{V}$, we need to show that $h^{\mathrm{inv}}(x)$ is in $\mathcal{W}$. Let $n\in\mathbb{N}, n\geq 1$ be arbitrary. By the above lemma, we know that there is $m_0\in\mathbb{N}$ such that $x_{m_0}\in B_1^{V,e}$, then $m_1\in\mathbb{N}, m_1>m_0$ with $x_{m_0}\succ x_{m_1}\in B_2^{V,e}$, etc., until we find $m_{n-1}>m_{n-2}$ with $x_{m_{n-2}}\succ x_{m_{n-1}}\in B_n^{V,e}$. Then the minimal grade sequence associated to $\overline{\overline{x}}(m_{n-1}+1)$ has at least length n. Therefore $h^{\mathrm{inv}}(\overline{\overline{x}}(m_{n-1}+1))$ is a sequence of length at least n. Thus we see that $h^{\mathrm{inv}}(x)$ is an infinite sequence of ever-shrinking basic dots of Baire space, and so a point in Baire space.

To show that $h^{\mathrm{inv}}(x)$ is also a point in $\mathcal{W}$, consider $a, c\in\mathbb{N}^*$ with $a\,\#_W c$. This by definition of $\#_W$ means that $h(a)\,\#_V h(c)$. Therefore $h(a)\,\#_V h(c)$ is one of the e-enumerated pairs of apart dots, say that $(h(a), h(c))=e_M$. By the above reasoning which demonstrated that $h^{\mathrm{inv}}(x)$ is a point of Baire space, we know that there is an $m\in\mathbb{N}$ such that the minimal grade sequence $p_0,\ldots,p_{j-1}$ associated to $\overline{\overline{x}}(m)$ has length $j = M+1$. By carefully looking at the construction of h^{inv} we see that $h^{\mathrm{inv}}(\overline{\overline{x}}(m))$ is a sequence $b_0, \ldots, b_M$ where $b_M \in B^{v,e}_{M+1}$ and so a fortiori $b_M \#_W a$ and/or $b_M \#_W c$.

Therefore $h^{\mathrm{inv}}(x)$ is also a point in $\mathcal{W}$. We leave it to the reader to verify that $h^{\mathrm{inv}} \circ h(x) \equiv x$ for all $x \in \mathcal{W}$ and $h \circ h^{\mathrm{inv}}(y) \equiv y$ for all $y \in \mathcal{V}$. (end of claim-proof) (END OF PROOF)

COROLLARY:

(i) Let $(\mathcal{V}, \mathcal{T}_\#)$ be a natural space, then there is a surjective $\preceq$-morphism from Baire space to $(\mathcal{V}, \mathcal{T}_\#)$. ('Baire space is a universal spread', 'every natural space is the natural image of Baire space', 'every natural space is a quotient topology of Baire space').

(ii) If $(\mathcal{V}, \mathcal{T}_\#)$ is a basic-open space (see definition 1.2.2) then $(\mathcal{V}, \mathcal{T}_\#)$ is isomorphic to a basic-open spread $(\mathcal{W}, \mathcal{T}_\#)$ whose tree is $(\mathbb{N}^*, \preceq_\omega)$.

A.3.5 Proof of theorem 2.2.1

That Cantor space is a universal fan is an easy and -in different terminology- well-known result, but we prove it anyway.

THEOREM: (repeated from 2.2.1) Let $(\mathcal{V}, \mathcal{T}_\#)$ be a fann, then there is a surjective morphism from Cantor space to $(\mathcal{V}, \mathcal{T}_\#)$.

PROOF: For each $a \in V$ we need to fix an element $x^a \in \mathcal{V}, x^a \preceq a$. For this we use a bijection $v : \mathbb{N} \to V$ with inverse v^{-1}. Now for a in V, let x^a be the unique point in $\mathcal{V}$ such that $x^a_0 = a$ and for each $n \in \mathbb{N}$ we have: $v^{-1}(x^a_{n+1}) = \mu s \in \mathbb{N}[v(s) \prec x^a_n]$.[14]

We define the surjective morphism f inductively. Determine the finite subset $\propto(\bigcirc_V) = \{a_i \,|\, i \leq m\} \subset V$ (in some enumeration, for some $m \in \mathbb{N}$). Let $n \in \mathbb{N}$ be such that $2^n < m \leq 2^{n+1}$. Putting ${}^{n+1}\{0,1\}^* = \{b \in \{0,1\}^* \,|\, \lg(b) = n+1\}$, we see that ${}^{n+1}\{0,1\}^*$ contains sufficient elements b_i (numbered in the obvious lexicographical/binary way) to put $f(b_i) = a_i$ for all $i \leq m$. For $m < i \leq 2^{n+1}$, we fix f by sending all of $\{y \in \mathcal{C} \,|\, y \preceq b_i\}$ to the fixed $x^{a_0} \in \mathcal{V}, x^{a_0} \preceq a_0$.

Now notice that V_{a_i} again determines a fann for all $i \leq m$, and so we can repeat this process for V_{a_i} and $\{y \in \mathcal{C} \,|\, y \preceq b_i\}$ which is isomorphic to $\mathcal{C}$. This inductively defines f. We leave it to the reader to verify that f is the desired surjective morphism. (END OF PROOF)

[14] This construction also works for spraids, not just for fanns.

A.3.6 Proof of theorem 3.1.0 We prove that for spraids the formal inductive covering relation $\blacktriangleleft$ equals the genetic inductive covering relation $\blacktriangleleft_\propto$.

THEOREM: (from 3.1.0) Let $(\mathcal{V}, \mathcal{T}_\#)$ be a spraid derived from $(V, \#, \preceq)$, and let $E, F \subseteq V$. Then $F \blacktriangleleft E$ iff $F \blacktriangleleft_\propto E$.

PROOF: We necessarily use both **PFI** and **PGI***. First let $F \blacktriangleleft_\propto E$, which means by definition that $\{a\} \blacktriangleleft_\propto E$ for all $a \in F$. Let $a \in F$, then $\{a\} \blacktriangleleft_\propto E$ which means E descends from a genetic bar G on V_a. We now show by genetic induction on G that $\{a\} \blacktriangleleft E$.

- $\mathbf{G}_\bigcirc$ If $G = \{\bigcirc_a\} = \{a\}$, then $a \in E_\preceq$, so by $\mathbf{Ind}_1^\blacktriangleleft$ and $\mathbf{Ind}_3^\blacktriangleleft$ we see that $\{a\} \blacktriangleleft E$.
- $\mathbf{G}_\propto$ Let $G = \bigcup_{b \in \propto(a)} B_b$ where for all $b \in \propto(a)$ we have that B_b is a genetic bar on B_b such that if E descends from B_b, then $\{b\} \blacktriangleleft E$. However, we do indeed know that E descends from B_b for all $b \in \propto(a)$ by the assumption on G. Therefore we find: $\{b\} \blacktriangleleft E$ for all $b \in \propto(a)$. This means that $\propto(a) \blacktriangleleft E$ by $\mathbf{Ind}_2^\blacktriangleleft$. On the other hand, one sees by $\mathbf{Ind}_1^\blacktriangleleft$ and $\mathbf{Ind}_2^\blacktriangleleft$ that $\{c \mid c \prec a\} \blacktriangleleft \propto(a)$. Combining this with $\mathbf{Ind}_5^\blacktriangleleft$ we obtain $\{a\} \blacktriangleleft \{c \mid c \prec a\} \blacktriangleleft \propto(a) \blacktriangleleft E$, and so by $\mathbf{Ind}_4^\blacktriangleleft$, we see that $\{a\} \blacktriangleleft E$.

Since a is arbitrary, this means that by $\mathbf{Ind}_2^\blacktriangleleft$ we can conclude $F \blacktriangleleft E$.

Now for the implication in the other direction, let $P(F, E)$ be the property: $F \blacktriangleleft_\propto E$. We prove that P satisfies $\mathbf{Ind}_1$ through $\mathbf{Ind}_5$ (for subsets F, E, D of V):

- $\mathbf{Ind}_1$ if $b \preceq c$ in V, then $\{b\} \blacktriangleleft_\propto \{c\}$ since $\{c\}$ descends from the genetic bar $\{\bigcirc_b\}$ on V_b.
- $\mathbf{Ind}_2$ if for all $a \in F$ we have $\{a\} \blacktriangleleft_\propto E$, then by definition of $\blacktriangleleft_\propto$, we have $F \blacktriangleleft_\propto E$.
- $\mathbf{Ind}_3$ if $F \blacktriangleleft_\propto D$ and $D \subseteq E$, this trivially implies that $F \blacktriangleleft_\propto E$.
- $\mathbf{Ind}_4$ suppose $F \blacktriangleleft_\propto D \blacktriangleleft_\propto E$. Let a in F, then $\{a\} \blacktriangleleft_\propto D \blacktriangleleft_\propto E$. We must show that $\{a\} \blacktriangleleft_\propto E$. Let G be a genetic bar on V_a such that D descends from G. To show that $\{a\} \blacktriangleleft_\propto E$ we use genetic induction and propositions 3.3.1 and A.3.8 to prove the statement: 'Let B be a genetic bar on V_a such that D descends from B, then $\{a\} \blacktriangleleft_\propto E$'.
  - $\mathbf{G}_\bigcirc$ If $B = \{\bigcirc_a\} = \{a\}$, then since D descends from B there is a $d \in D$ with $a \preceq d$. Since $\{d\} \blacktriangleleft_\propto E$ we find a genetic bar H on V_d such that E descends from H. Now by proposition 3.3.1, the reduction $H^{\upharpoonright a}$ of H to V_a contains a genetic bar H' on V_a. Clearly E descends from H' as well, showing that $\{a\} \blacktriangleleft_\propto E$.

G$_\propto$ Let $B = \bigcup_{b \in \propto(a)} B_b$ where for each $b \in \propto(a)$ we have that B_b is a genetic bar on V_b satisfying: if D descends from B_b then $\{b\} \blacktriangleleft_\propto E$. Still, we already know that D descends from B_b for each $b \in \propto(a)$. Therefore for each $b \in \propto(a)$ we know $\{b\} \blacktriangleleft_\propto E$. This means that for each $b \in \propto(a)$ we have a genetic bar H_b on V_b such that E descends from H_b. Now by proposition A.3.8 we see that $H = \bigcup_{b \in \propto(a)} H_b$ is a genetic bar on V_a such that E descends from H, showing that $\{a\} \blacktriangleleft_\propto E$.

By **PGI*** our statement is proven, and we conclude (since G is a genetic bar on V_a such that D descends from G) that $\{a\} \blacktriangleleft_\propto E$. Since $a \in F$ is arbitrary, this shows that $F \blacktriangleleft_\propto E$.

Ind$_5$ $b \blacktriangleleft_\propto \{d \mid d \prec b\}$, since $\{d \mid d \prec b\}$ contains the genetic bar $\propto(b)$ on V_b.

By **PFI** we now conclude that $F \blacktriangleleft E$ implies $F \blacktriangleleft_\propto E$. (END OF PROOF)

A.3.7 Proof of proposition 3.3.0

We wish to show that for a spraid $(\mathcal{V}, \mathcal{T}_\#)$ derived from $(V, \#, \preceq)$, genetic bars on V correspond to genetic bars on $V^\heartsuit$ in a precise way. We could call this the unglueing of genetic bars on V. From this correspondence it follows that trail morphisms are inductive iff they are inductive as refinement morphism.

LEMMA: Let $(\mathcal{V}, \mathcal{T}_\#)$ be a spraid derived from $(V, \#, \preceq)$. Let $c \in V, d \in V^\heartsuit$ and let $G \subseteq V$ be a genetic bar on V_c and H a genetic bar on $V^\heartsuit_d$. Then:

(i) $\mathbf{id}_*(H)$ is a genetic bar on $V_{\mathbf{id}_*(d)}$.

(ii) for all $c' \in V^\heartsuit$ with $\mathbf{id}_*(c') = c$ there is a genetic bar G' on $V^\heartsuit_{c'}$ such that $\mathbf{id}_*(G') = G$.

COROLLARY: There is a direct correspondence between genetic bars on V and genetic bars on $V^\heartsuit$.

PROOF: Ad (i): by genetic induction:

G$_\bigcirc$ If $H = \{\bigcirc_d\} = \{d\}$, then $\mathbf{id}_*(H) = \{\mathbf{id}_*(d)\} = \{\bigcirc_{\mathbf{id}_*(d)}\}$, and we are done.

G$_\propto$ Let $H = \bigcup_{b \in \propto(d)} B_b$ where for each $b \in \propto(d)$ we know that $\mathbf{id}_*(B_b)$ is a genetic bar on $V_{\mathbf{id}_*(b)}$. Notice that $\mathbf{id}_*(\propto(d)) = \propto(\mathbf{id}_*(d))$ in a trivial bijective correspondence. Put $b' = \mathbf{id}_*(b)$ for $b \in \propto(d)$ and $B'_{b'} = \mathbf{id}_*(B_b)$, then $\mathbf{id}_*(H) = \bigcup_{b' \in \propto(\mathbf{id}_*(d))} B'_{b'}$. This shows $\mathbf{id}_*(H)$ is a genetic bar on $V_{\mathbf{id}_*(d)}$, and we are done.

Ad (ii): also by genetic induction. Take any $c' \in V^{\between}$ such that $\mathbf{id}_*(c') = c$. We again use the bijective correspondence between $\propto(c') \subset V^{\between}$ and $\propto(c) \subset V$, but now for $b \in \propto(c) \subset V$ we let $b' \in \propto(c') \subset V^{\between}$ be such that $\mathbf{id}_*(b') = b$.

- $\mathbf{G}_{\bigcirc}$ If $G = \{\bigcirc_c\}$, then take $G' = \{\bigcirc_{c'}\}$.
- $\mathbf{G}_{\propto}$ Let $G = \bigcup_{b \in \propto(c)} B_b$ where for each $b \in \propto(c)$ we have a genetic bar $B'_{b'}$ on $V^{\between}_{b'}$ such that $\mathbf{id}_*(B'_{b'})$ is B_b. Now take $G' = \bigcup_{b' \in \propto(c')} B'_{b'}$, and we are done.

(END OF PROOF)

PROPOSITION: (from 3.3.0) Let g be a $\wr$-morphism from $(\mathcal{V}, \mathcal{T}_{\#})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$. Then g is an inductive $\wr$-morphism iff g is inductive as a $\preceq$-morphism from $(\mathcal{V}^{\between}, \mathcal{T}^{\wr}_{\#})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$.

PROOF: With the above lemma, the proof now is trivial. (END OF PROOF)

A.3.8 Proof of lemma 3.3.2 The proof of lemma 3.3.2 is quite involved. We even need an extra proposition, which has intrinsic value since it can be put to use often in genetic induction proofs:

PROPOSITION: Let $(\mathcal{V}, \mathcal{T}_{\#})$ be a spraid derived from the pre-natural space $(V, \#, \preceq)$. Let $a \in V$ and suppose G is a genetic bar on V_a where for all $e \in G$ we have a genetic bar D_e on V_e. Then $D = \bigcup_{e \in G} D_e$ is a genetic bar on V_a.

PROOF: By genetic induction on G:

- $\mathbf{G}_{\bigcirc}$ If $G = \{\bigcirc_a\} = \{a\}$ then we are trivially done.
- $\mathbf{G}_{\propto}$ Let $G = \bigcup_{b \in \propto(a)} B_b$ where for all $b \in \propto(a)$: if for all $e \in B_b$ there is a genetic bar D_e on V_e, then $D^b = \bigcup_{e \in B_b} D_e$ is a genetic bar on V_b. However, we already know for all $b \in \propto(a)$ that for all $e \in B_b$ there is a genetic bar D_e on V_e and so D^b is a genetic bar on V_b. Therefore $D = \bigcup_{b \in \propto(a)} D^b$ is a genetic bar on V_a.

(END OF PROOF)

LEMMA: (repeated from 3.3.2) Let f be an inductive morphism between the two spraids $(\mathcal{V}, \mathcal{T}_{\#_1})$ and $(\mathcal{W}, \mathcal{T}_{\#_2})$, derived from $(V, \#_1, \preceq_1)$ and $(W, \#_2, \preceq_2)$ respectively. Let $a \in W$, and let G be a genetic bar on W_a. Then for all $d \in V$: if $f(d) \in W_a$, then $\overleftarrow{f}(G)$ contains a genetic bar on V_d.

PROOF: By genetic induction on the bar G on W_a.

G$_{\bigcirc}$ $G=\{\bigcirc_a\}=\{a\}$. Let $d\in V$ with $f(d)\in W_a$. Then $d\in\overleftarrow{f}(a)=\overleftarrow{f}(G)$ so $\{\bigcirc_d\}$ is a genetic bar on V_d contained in $\overleftarrow{f}(G)$.

G$_{\propto}$ $G=\bigcup_{b\in\propto(a)}B_b$ and for all $b\in\propto(a)$ we know that for all $d\in V$: if $f(d)\in W_b$, then $\overleftarrow{f}(B_b)$ contains a genetic bar on V_d (the induction property).

Now let $d\in V$ with $f(d)\in W_a$. If $f(d)\prec a$, then there is $b\in\propto(a)$ with $f(d)\in W_b$ which by the induction premise implies that there is a genetic bar on V_d which is contained in $\overleftarrow{f}(B_b)\subseteq\overleftarrow{f}(G)$ and we are done.

Else, $f(d)=a$. Then we consider the genetic bar $H=\propto(a)$ on W_a. By proposition 3.3.1 the expansion $H^{\downarrow\circ}$ of H to W is a genetic bar on W, so $\overleftarrow{f}(H^{\downarrow\circ})$ contains a genetic bar E on V. Again by proposition 3.3.1, the reduction $E^{\uparrow d}$ of E to V_d contains a genetic bar D on V_d.

By construction of E, we see that for $e\in D$ there is a $b\in\propto(a)$ such that $f(e)\in W_b$. Therefore by the induction assumption, $\overleftarrow{f}(B_b)$ contains a genetic bar K_e on V_e. Now by the proposition above, $K=\bigcup_{e\in D}K_e$ is a genetic bar on V_d contained in $\overleftarrow{f}(G)$.

(END OF PROOF)

A.3.9 Proof of proposition and lemma 3.3.3

In formal topology (see [Pal2005]) a continuous^BIS function from $\mathbb{R}$ to $\mathbb{R}$ is representable by a formal mapping from the formal reals to the formal reals, and vice versa each such mapping represents a continuous^BIS function. We repeat this insight in our setting (recall for the proof that ${}^s\sigma_{\mathbb{R}}^{\between}=\{a\in\sigma_{\mathbb{R}}^{\between}\,|\,\lg(a)=s\}$):

PROPOSITION: Let f be a continuous^BIS function from $\mathbb{R}$ to $\mathbb{R}$. Then there is an inductive morphism f^* from $\sigma_{\mathbb{R}}$ to $\sigma_{\mathbb{R}}$ such that for all $x\in\sigma_{\mathbb{R}}$ we have $f(x)\equiv f^*(x)$ (where we identify $\mathbb{R}$ and $\sigma_{\mathbb{R}}$ for convenience). Conversely, if g is an inductive morphism from $\sigma_{\mathbb{R}}$ to $\sigma_{\mathbb{R}}$, then as a function g is uniformly continuous on each compact subspace of $\mathbb{R}$.

PROOF: We first create f^* as a trail morphism, since then we need to keep only little track of the intersection properties of intervals. Therefore we use the unglueing $\sigma_{\mathbb{R}}^{\between}$ of $\sigma_{\mathbb{R}}$, which was defined in 2.2.0.

We turn to the uniform-continuity properties of f on compact subsets of $\mathbb{R}$. We divide $\mathbb{R}$ in the pairwise overlapping compact spaces $([m,m+2])_{m\in\mathbb{Z}}$, and by the uniform-continuity properties of f (using $\mathbf{AC}_{00}$) we can determine a sequence $(\delta(m,n))_{m\in\mathbb{Z},n\in\mathbb{N}}$ of elements of $\mathbb{N}$ (with $\delta(m,n)<\delta(m,n+1)$) such that if $a=[\frac{k}{2^{\delta(m,n)}},\frac{k+2}{2^{\delta(m,n)}}]$ is a subinterval of $[m,m+2]$, then $f(a)\subset[\frac{s}{2^n},\frac{s+2}{2^n}]$ for some $s\in\mathbb{Z}$.

Again using $\mathbf{AC}_{00}$, we can turn this into a function $\tilde{f}$ on all pairs (m,a) where $a=[\frac{k}{2^{\delta(m,n)}},\frac{k+2}{2^{\delta(m,n)}}]$ is a subinterval of $[m,m+2]$. We hereby define $\tilde{f}$ in such a way that $\tilde{f}((m,a))=[\frac{s}{2^n},\frac{s+2}{2^n}]$ for some $s\in\mathbb{Z}$ and $f(a)\subset\tilde{f}((m,a))$ for all such pairs (m,a).

This will yield the definition of the desired morphism f^* on basic dots a, by induction on $t=\lg(a)$. For $t=0$ put $f^*(\bigcirc^*)\underset{D}{=}\bigcirc_{\mathbb{R}}$. Next, suppose f^* has been defined on ${}^s\sigma^{♉}_{\mathbb{R}}$ for $s<t>0$, and let $a=a_0,\ldots,a_{t-1}\in{}^t\sigma^{♉}_{\mathbb{R}}$. Determine $m\in\mathbb{Z}$ with $a_0=[m,m+2]$ and the smallest $n\in\mathbb{N}$ such that $t=\lg(a)<\delta(m,n)+1$.

If $n=0$, this means that for any $c\,{}^*\!\succeq a_{t-1}$, in the process of defining $\tilde{f}$ we did not assign to (m,c) an interval b such that $f(c)\subset b$, because even a_{t-1} is still too large. Therefore we put $f^*(a)\underset{D}{=}\bigcirc_{\mathbb{R}}$.
If on the other hand $n>0$, then $t-1\geq\delta(m,n-1)$. Putting $s=\delta(m,n-1)$, we define: $f^*(a)\underset{D}{=}\tilde{f}(m,a_s)\cap f^*(a_0,\ldots,a_{s-1})$, interpreted as intervals.

We leave it to the reader to verify that f^* is a $\preceq$-morphism from $\sigma^{♉}_{\mathbb{R}}$ to $\sigma_{\mathbb{R}}$ (and therefore a $\wr$-morphism from $\sigma_{\mathbb{R}}$ to itself) which represents f.

To finish the proof of the first part of the theorem, we need to show that f^* is an inductive morphism. For this, let G be a genetic bar on $\sigma_{\mathbb{R}}$. We need to show that $\overleftarrow{f^*}(G)$ contains a genetic bar on $\sigma^{♉}_{\mathbb{R}}$. If $G=\{\bigcirc_{\mathbb{R}}\}$ then we are trivially done. Else consider $\overleftarrow{f^*}(G)$ on the subfan $\rho^*_{[m,m+2]}=\{a\in\sigma^{♉}_{\mathbb{R}}\,|\,a\preceq^*[m,m+2]\}$ for given $m\in\mathbb{N}$. By the uniform continuity of f, we find $N\in\mathbb{N}$ such that $f([m,m+2])\subset[-N+1,N-1]$. Also, $[-N,N]$ determines a subfann $\tau=\tau_{[-N,N]}$ of $\sigma_{\mathbb{R}}$, and we know that $G_\tau=G\cap\tau$ is finite by theorem 3.2.0.

And so we find a least $M\in\mathbb{N}$ such that G_τ descends from $\{a\in\tau\,|\,\lg(a)=M+1\}$, in other words such that the intervals of the type $[\frac{k}{2^M},\frac{k+2}{2^M}]$ form a refinement of G_τ. Now on $\rho^*_{[m,m+2]}$ we see that for each element b of the genetic bar $H_{G,[m,m+2]}=\{a\in\rho^*_{[m,m+2]}\mid\lg(a)=\delta(m,M)+1\}$, there is a $c\in G_\tau$ such that $f^*(b)\preceq c$. We conclude: $H_{G,[m,m+2]}$ is contained in $\overleftarrow{f^*}(G)$. Thus we canonically obtain a sequence of genetic bars $(H_{G,[m,m+2]})_{m\in\mathbb{Z}}$ (each on the respective $\rho^*_{[m,m+2]}$) such that $H_{G,[m,m+2]}$ is contained in $\overleftarrow{f^*}(G)$ for each $m\in\mathbb{Z}$.

It suffices to consider that by definition of genetic bars, $H=\bigcup_{m\in\mathbb{Z}}H_{G,[m,m+2]}$ is a genetic bar on $\sigma^{♉}_{\mathbb{R}}$ (trivially H is contained in $\overleftarrow{f^*}(G)$). This finishes the proof of the first half of the theorem.

For the second half of the theorem, let g be an inductive morphism from $\sigma_{\mathbb{R}}$ to $\sigma_{\mathbb{R}}$. Copying from the reasoning and constructions above, we look at the genetic bar $G_n = \{\alpha \in \sigma_{\mathbb{R}} \,|\, \lg(\alpha) = n+1\}$, in other words the intervals of the form $[\frac{k}{2^n}, \frac{k+2}{2^n}]$. We see that $\overleftarrow{g}(G_n)$ contains a genetic bar H_n, which is finite on each subfan of $\sigma_{\mathbb{R}}$. This shows that for a subfan τ and any $n \in \mathbb{N}$, there is an $m \in \mathbb{N}$ such that for all $x, y \in \tau$ with $d_{\mathbb{R}}(x,y) \leq 2^{-m}$ we have $d_{\mathbb{R}}(g(x), g(y)) \leq 2^{-n}$. A BISH-compact subspace X is always contained in a subfan τ of $\sigma_{\mathbb{R}}$, so we see that g is indeed uniformly continuous on any BISH-compact subspace of $\mathbb{R}$. (END OF PROOF)

REMARK: Proposition 2.3.2 (and its proof) shows how to represent f^* as a $\preceq$-morphism from $\sigma_{\mathbb{R}}$ to $\sigma_{\mathbb{R}}$. (END OF REMARK)

We also need to prove our remark that the situation is very different when we replace the image space with $\mathbb{R}^+$.

LEMMA: The statement that every uniformly continuous function from $[0,1]$ to $\mathbb{R}^+$ is representable by an inductive morphism from $\sigma_{[0,1]}$ to $\sigma_{\mathbb{R}^+}$ is equivalent to the fan theorem **FT**.

PROOF: The proof uses the equivalence of **FT** to the statement that each uniformly continuous function from $[0,1]$ to $\mathbb{R}^+$ is bounded away from 0, which was already proved in [Jul&Ric1984].

First let f be a uniformly continuous function from $[0,1]$ to $\mathbb{R}^+$ which is representable by an inductive morphism f^* from $\sigma_{[0,1]}$ to $\sigma_{\mathbb{R}^+}$. Notice that $G = \propto(\bigcirc_{\mathbb{R}})$ is a genetic bar on $\sigma_{\mathbb{R}^+}$, therefore $\overleftarrow{f^*}(G)$ contains a (finite) genetic bar H on $\sigma_{[0,1]}$. Looking at the finite set $f^*(H) \preceq G$ we conclude that there is $n \in \mathbb{N}$ such that $f(x) > 2^{-n}$ for all $x \in [0,1]$, in other words f is bounded away from 0.

Conversely, suppose f is bounded away from 0, in other words there is $n \in \mathbb{N}$ such that $f(x) > 2^{-n}$ for all $x \in [0,1]$. By the uniform continuity of f, we can now proceed almost exactly as in the proof of the proposition above to construct an inductive morphism f^* from $\sigma_{[0,1]}$ to $\sigma_{\mathbb{R}^+}$ representing f. We leave this to the reader.

To finish the proof, we now invoke the well-known result that **FT** is equivalent to the statement that each uniformly continuous function from $[0,1]$ to $\mathbb{R}^+$ is bounded away from 0 (see [Jul&Ric1984], or for another elegant proof, see proposition 4.2 in [Waa2005]). (END OF PROOF)

A.3.10 Proof of proposition 3.4.1

For this we first need a lemma on genetic and inductive bars:

LEMMA: Let $(\mathcal{V}, \mathcal{T}_{\#})$ be a spraid derived from the pre-natural space $(V, \#, \preceq)$, and let D_0, D_1 be (i) genetic (ii) inductive bars on V. Then:

(i) $\min(D_0, D_1) = \{d \in V \mid \exists c \in V \exists i \in \{0,1\} [d \in D_i \wedge c \in D_{1-i} \wedge d \preceq c]\}$ is a genetic bar on V.

(ii) $(D_0)_{\preceq} \cap (D_1)_{\preceq}$ is an inductive bar on V.

PROOF: For genetic bars D_0, D_1 we prove (i) by genetic induction on D_0:

G$_{\bigcirc}$ If $D_0 = \{\bigcirc\}$, then $\min(D_0, D_1) = D_1$ and we are done.

G$_{\propto}$ Else $D_0 = \bigcup_{b \in \propto(\bigcirc)} B_b$ where for all $b \in \propto(\bigcirc)$ we have that $\min(B_b, D_1)$ is a genetic bar on V_b. Then $\min(D_0, D_1) = \bigcup_{b \in \propto(\bigcirc)} \min(B_b, D_1)$ is also a genetic bar.

Ad(ii): for inductive bars D_0, D_1 descending from genetic bars G_0, G_1 respectively, it suffices to see that $(D_0)_{\preceq} \cap (D_1)_{\preceq}$ contains $\min(G_0, G_1)$, which is a genetic bar by (i). For let $d \in \min(G_0, G_1)$. Without loss of generality $d \in G_1$ and there is $d \preceq c \in G_0$. Then we can determine $d \preceq b \in D_1$ and $c \preceq a \in D_0$, whence $d \preceq c \preceq a$ and so $d \in (D_0)_{\preceq} \cap (D_1)_{\preceq}$. (END OF PROOF)

COROLLARY: By induction on n we can now conclude: if $B_0, \ldots, B_n$ is a finite sequence of (i) genetic (ii) monotone inductive bars, then

(i) $\min(B_0, \ldots, B_n) = \{d \in V \mid \exists i \leq n [d \in B_i \wedge \forall j \leq n, j \neq i \exists c \in B_j [d \preceq c]\}$ is a genetic bar on V.

(ii) $\bigcap_{i \leq n} B_i$ is a monotone inductive bar.

REMARK: An interesting exercise for the reader is to see why for inductive bars D_0, D_1 descending from genetic bars G_0, G_1 respectively, the bar $\min(D_0, D_1)$ does not necessarily descend from $\min(G_0, G_1)$. For a spread $(\mathcal{V}, \mathcal{T}_{\#})$ however $\min(D_0, D_1)$ does descend from $\min(G_0, G_1)$. (END OF REMARK)

PROPOSITION: (repeated from 3.4.1)

(I) For a spraid $(\mathcal{V}, \mathcal{T}_{\#})$ with corresponding pre-natural space $(V, \#, \preceq)$, the collection $\mathcal{T}_{\#}^{\propto}$ is a topology which is refined by $\mathcal{T}_{\#}$.

(II) Let $(\mathcal{V}, \mathcal{T}_{\#})$ be an inductive spraid with corresponding pre-natural space $(V, \#, \preceq)$. Then for finite subsets $A \# B$ of V, the subset $C = \{c \in V \mid c \# A \vee c \# B\}$ is an inductive bar on $(\mathcal{V}, \mathcal{T}_{\#})$.

PROOF: Ad (I): That $\mathcal{T}_\#$ refines $\mathcal{T}_\#^\propto$ is trivial. To show that $\mathcal{T}_\#^\propto$ is a topology, we check the definitions:

Top₁ Trivially the empty set $\emptyset$ and $\mathcal{V}$ are in $\mathcal{T}_\#^\propto$ (for any $x \in \mathcal{V}$ the set $B^x_{\mathcal{V}} = V$ is an inductive bar).

Top₂ Let $\mathcal{U}, \mathcal{W} \in \mathcal{T}_\#^\propto$, we must show that $\mathcal{U} \cap \mathcal{W} \in \mathcal{T}_\#^\propto$. For this, let $x \in \mathcal{U} \cap \mathcal{W}$, and consider that $B^x_{\mathcal{U}\cap\mathcal{W}} = B^x_{\mathcal{U}} \cap B^x_{\mathcal{W}}$ since for all $b \in B^x_{\mathcal{U}} \cap B^x_{\mathcal{W}}$ we can decide: case $\mathcal{U}(1)$ $b \# x_{\lg(b)}$ (done) or case $\mathcal{U}(2)$ $[b] \subseteq \mathcal{U}$, in which case we consider case $\mathcal{W}(1)$ $b \# x_{\lg(b)}$ (done) or case $\mathcal{W}(2)$ $[b] \subseteq \mathcal{W}$ whence we see that $[b] \subseteq \mathcal{U} \cap \mathcal{W}$ and we are done too. The bars $B^x_{\mathcal{U}}$ and $B^x_{\mathcal{W}}$ are monotone and inductive, and so $B^x_{\mathcal{U}\cap\mathcal{W}} = B^x_{\mathcal{U}} \cap B^x_{\mathcal{W}}$ is a monotone inductive bar also by (ii) of the above corollary.

Top₃ Let $\mathcal{U} \subseteq \mathcal{V}$ be such that for all $x \in \mathcal{U}$ there is a $\mathcal{W} \ni x$ in $\mathcal{T}_\#^\propto$ such that $\mathcal{W} \subseteq \mathcal{U}$. Let $x \in \mathcal{U}$, determine $\mathcal{W} \ni x$ in $\mathcal{T}_\#^\propto$ such that $\mathcal{W} \subseteq \mathcal{U}$. Then clearly $B^x_{\mathcal{W}} \subseteq B^x_{\mathcal{U}}$ therefore $B^x_{\mathcal{U}}$ is an inductive bar, which since x is arbitrary shows that $\mathcal{U}$ is in $\mathcal{T}_\#^\propto$.

Ad (II): First suppose either A or B is empty, then the conclusion is trivially fulfilled. Else, let $A = \{a_0, \ldots, a_n\}$ and $B = \{b_0, \ldots, b_m\}$ with $a_i \# b_j$ for all $i \leq n, j \leq m$. By definition of 'inductive spraid', for any $i \leq n, j \leq m$ the set $D_{i,j} = \{d \in V \mid d \# a_i \vee d \# b_j\}$ is an inductive bar (which is also clearly monotone). Therefore by (ii) of the above corollary, $D = \bigcap_{i \leq n, j \leq m} D_{i,j}$ is a monotone inductive bar. However, D actually equals $C = \{c \in V \mid c \# A \vee c \# B\}$, therefore C is an inductive bar as promised. (END OF PROOF)

A.3.11 Proof of meta-theorem 3.4.0

In paragraph 3.4.0 we indicated pointwise problems regarding inductive morphisms in BISH:

META-THEOREM: (repeated from 3.4.0)
In RUSS (and by implication BISH) we have the following problems regarding pointwise use of inductive definitions:

P_1 Uniform continuity of a function f does not imply that there is an inductive morphism representing f. Counterexamples can be given even for uniformly continuous functions from $[0, 1]$ to $\mathbb{R}^+$. In other words: uniform continuity does not imply inductive representability.

P_2 Weak completeness[15] of a compact space is not preserved under induc-

[15] The property for a located subset A of a metric space (X, d) that for all $x \in X$: if $x \# a$ for all $a \in A$, then $d(x, A) = \inf(\{d(x, a) \mid a \in A\}) > 0$.

tivity. In RUSS, even for an inductive morphism from $[0,1]$ to $[0,1]$, the image of a compact subspace may be strongly incomplete.

P_3 Inductive representability is not preserved under the restriction of a function to its pointwise image space. This follows from the counterexamples for P_1, since every uniformly continuous function from $[0,1]$ to $\mathbb{R}$ is inductively representable by proposition 3.3.3. Therefore we can expect problems with the reciprocal function $x \to \frac{1}{x}$, and must continually address these problems by adapting our definitions.

Therefore in BISH, the desirable properties associated with the problems above cannot be shown to hold without further assumptions. In fact, assertion of any of these properties implies the fan theorem **FT**.

PROOF: The proof is derived from the construction of ContraCantor space, which is a compact subspace $Ↄ_{[0,1]}$ of $[0,1]$ such that if we write $\mathcal{C}_{[0,1]}$ for the standard embedding of Cantor space in $[0,1]$, we see: $d_{\mathbb{R}}(Ↄ_{[0,1]}, \mathcal{C}_{[0,1]}) = 0$ and yet in RUSS we also have $d_{\mathbb{R}}(x, \mathcal{C}_{[0,1]}) > 0$ for $\forall x \in Ↄ_{[0,1]}$. ContraCantor space is defined in the examples' section of the appendix A.2.3 using the Kleene Tree, and the properties above are proved in proposition A.2.3.

To show P_1, consider the uniformly continuous function $d_{\mathcal{C}}$ from $[0,1]$ to $[0, \frac{1}{6}]$ given by $d_{\mathcal{C}}(x) = d_{\mathbb{R}}(x, \mathcal{C}_{[0,1]})$.[16] The restriction of $d_{\mathcal{C}}$ to $Ↄ_{[0,1]}$ is an example in RUSS of a uniformly continuous function (from $Ↄ_{[0,1]}$ to $\mathbb{R}^+$) which cannot be represented as an inductive morphism from the subfann $Ↄ_{[0,1]}$ to $\mathbb{R}^+_{\mathrm{nat}}$. Notice that $d_{\mathcal{C}}$ is a beautiful function, and that $Ↄ_{[0,1]}$ is a beautiful compact space, so this type of problem can be expected to crop up at any time.

The above example also shows what we mean with P_3, since $d_{\mathcal{C}}$ seen as a uniformly continuous funtion from $Ↄ_{[0,1]}$ to $\mathbb{R}$ is representable as an inductive morphism from the subfann $Ↄ_{[0,1]}$ to $\mathbb{R}_{\mathrm{nat}}$ (see prp. 3.3.3 and A.3.9).

To show P_2, we consider the subset $d_{\mathcal{C}}(Ↄ_{[0,1]})$ of $[0, \frac{1}{6}]$, which is obviously strongly incomplete, since $0 \# d_{\mathcal{C}}(Ↄ_{[0,1]})$ and yet $d_{\mathbb{R}}(Ↄ_{[0,1]}, \mathcal{C}_{[0,1]}) = 0$.

Another example in RUSS of a strongly incomplete fann-image under an inductive morphism was already given in [Waa2005]. This example consists of $[0,1]^{\mathbb{N}}_{\mathrm{bin}}$ and a recursive β in $[0,1]^{\mathbb{N}}$ such that $d_{\mathbb{R}^{\mathbb{N}}}(\beta, [0,1]^{\mathbb{N}}_{\mathrm{bin}}) = 0)$ and yet $\beta \#_{\mathbb{R}^{\mathbb{N}}} x$ for all recursive $x \in [0,1]^{\mathbb{N}}_{\mathrm{bin}}$. (END OF PROOF)

[16] The distance of a point to a compact subspace can always be constructively calculated.

A.3.12 Proof of theorem 3.4.3

We prove that every complete metric space is homeomorphic to a $\propto$-spread, by adapting our proof in 1.2.3 that every complete metric space is homeomorphic to a natural space.

THEOREM: (from 3.4.3) Every complete metric space (X, d) is homeomorphic to a $\propto$-spraid.

PROOF: For a complete metric space (X, d) with dense subset $(a_n)_{n\in\mathbb{N}}$, we constructed a homeomorphic $(V, \#, \preceq)$ where $V = \{B(a_n, 2^{-s}) \mid n, s\in\mathbb{N}\}$. If we look more carefully, we see that the trail space of this $(\mathcal{V}, \mathcal{T}_\#)$ contains a (homeomorphic) $\propto$-spread.

We copy the notation in A.3.2. We defined $\preceq$ and $\#$ on pairs of basic dots in V such that $B(a_n, 2^{-s}) \prec B(a_m, 2^{-t})$ implies $s > t$ and $d(a_n, a_m) < 2^{-t} - 2^{-s}$, and secondly $B(a_n, 2^{-s}) \# B(a_m, 2^{-t})$ implies $d(a_n, a_m) > 2^{-s} + 2^{-t} + 2^{-s-t-1}$.

We now define new basic dots, similar to forming the trail space (see 1.1.4). Firstly, for a basic dot $a = B(a_n, 2^{-s}) \in V$ we put $\mathrm{gd}(a) \underset{D}{=} s$.
Let p be a point in $\mathcal{V}$, then remember we write $\overline{p}(n)$ for the finite sequence $p_0, \ldots, p_{n-1}$ of basic dots in V. Notice that by definition $p_0 \succeq \ldots \succeq p_{n-1}$. A finite sequence $a = a_0 \succeq \ldots \succeq a_{n-1}$ of basic dots in V is called a *graded trail* in $(V, \preceq)$ (of length n) iff $\mathrm{gd}(a_i) = i$ for $0 \leq i < n$. The empty sequence is the unique graded trail of length 0, and denoted $\bigcirc^*$. The countable set of graded trails in $(V, \preceq)$ is denoted $V^{\wr}_{\mathrm{gd}}$, notice that $V^{\wr}_{\mathrm{gd}} \subset V^{\wr} = \{\overline{p}(n) \mid n\in\mathbb{N}, p\in\mathcal{V}\}$.

The pre-natural space $(V^{\wr}_{\mathrm{gd}}, \preceq^*, \#^*)$ (see def. 1.1.4) induces a spread, which we call $\mathcal{V}^{\wr}_{\mathrm{gd}}$. We show that $\mathcal{V}^{\wr}_{\mathrm{gd}}$ is an inductive spread by two claims:

claim For $c \#^* d \in V^{\wr}_{\mathrm{gd}}$ there is a genetic bar B on $V^{\wr}_{\mathrm{gd}}$ such that for all $b\in B$ we have: $(b \#^* c \vee b \#^* d)$.

proof Let $c = c_0, \ldots, c_s$ and $d = d_0, \ldots, d_t$, with $c_s = B(a_n, 2^{-s})$, $d_t = B(a_m, 2^{-t})$ for certain $n, m, s, t\in\mathbb{N}$. By definition $c \#^* d$ means $B(a_n, 2^{-s}) \# B(a_m, 2^{-t})$.

By the properties of $\#$ for $(\mathcal{V}, \mathcal{T}_\#)$ (see above), if $B(a_n, 2^{-s}) \# B(a_m, 2^{-t})$, then $d(a_n, a_m) > 2^{-s} + 2^{-t} + 2^{-s-t-1}$. This means that for any basic dot e in V with $\mathrm{gd}(e) \geq s+t+3$ we can decide: $e \# B(a_n, 2^{-s})$ or $e \# B(a_m, 2^{-t})$, by looking at $e = B(a_k, 2^{-\mathrm{gd}(e)})$ to see that either $d(a_k, a_n)$ or $d(a_k, a_m)$ is big enough. We conclude that the genetic bar $B = \{b\in V^{\wr}_{\mathrm{gd}} \mid \lg(b) = s+t+3\}$ satisfies the requirements. (end of claim-proof)

claim Let $\mathcal{U}$ be open in $\mathcal{V}^{\wr}_{\text{gd}}$, and $x \in \mathcal{U}$. Then $B^x_{\mathcal{U}} = \{b \in V^{\wr}_{\text{gd}} \mid b \#^* x_{\lg(b)} \vee [b] \subseteq \mathcal{U}\}$ is an inductive bar on $V^{\wr}_{\text{gd}}$.

proof (Remember the definition of $\mathbf{id}_*$ in 1.1.4.) Looking at x, we see that there are $a \in V$ and $t, n, i \in \mathbb{N}, i \geq 1$ such that $[x_t] \subseteq \mathcal{U}$ and $\mathbf{id}_*(x_t) = B(a_n, 2^{-i+1})$. Determine $j, m \in \mathbb{N}$ such that $\mathbf{id}_*(x_j) \preceq B(a_m, 2^{-i}) \preceq B(a_n, 2^{-i+1})$. By definition of $\preceq$ (see A.3.2), we have: $d(a_n, a_m) < 2^{-i+1} - 2^{-i} = 2^{-i}$. Determine $s \in \mathbb{N}$ such that $d(a_n, a_m) < 2^{-i} - 2^{-s+2}$. Determine $u \in \mathbb{N}$ such that $\text{gd}(\mathbf{id}_*(x_u)) \geq s$.

Now let $e \in V$ with $\text{gd}(e) = s$, say $e = B(a_k, 2^{-s})$. By the properties above, we can determine: $e \# \mathbf{id}_*(x_u)$ or $e = B(a_k, 2^{-s}) \subseteq B(a_n, 2^{-i+1}) = \mathbf{id}_*(x_t)$. But then in turn for any $b \in V^{\wr}_{\text{gd}}, \lg(b) \geq \max(s, u)$ we can determine: $b \#^* x_{\lg(b)}$ or $[b] \subseteq \mathcal{U}$. So $B^x_{\mathcal{U}}$ contains the genetic bar $\{b \in V^{\wr}_{\text{gd}} \mid \lg(b) = \max(s, u)\}$, and is therefore inductive. (end of claim-proof)

By the two above claims, we see that $\mathcal{V}^{\wr}_{\text{gd}}$ is a $\propto$-spread. We leave it to the reader to verify that $(\mathcal{V}^{\wr}_{\text{gd}}, \mathcal{T}^{\wr}_{\#})$ is homeomorphic to (X, d). (END OF PROOF)

REMARK: The question which representation to choose for complete metric spaces can probably not be answered in just one way. We believe that detailed study of this question, for various spaces, yields both theoretical and practical advantages (from the APPLIED perspective). (END OF REMARK)

A.3.13 Proof of proposition 3.5.2 (iv) We prove that an (in)finite product of star-finite spreads (see def. 4.0.10) is faithful in CLASS and INT. For the proof we will use some theory developed in the proof of theorem 4.0.8, in paragraph A.3.18.

PROPOSITION: (CLASS, INT, from 3.5.2) Let $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n \in \mathbb{N}}$ be star-finite spraids derived from the corresponding pre-natural $((V_n, \#_n, \preceq_n))_{n \in \mathbb{N}}$. Then the finite spraid-products $\Pi^{\sigma}_{i \leq n}(\mathcal{V}_i, \mathcal{T}_{\#_i})$ (for $n \in \mathbb{N}$) and the infinite spraid-product $\Pi^{\sigma}_{n \in \mathbb{N}}(\mathcal{V}_n, \mathcal{T}_{\#_n})$ are faithful.

PROOF: We only prove the infinite-product case, the finite-product case is completely similar and easier. Let $\mathcal{U}$ be open in $\Pi^{\sigma}_{n \in \mathbb{N}}(\mathcal{V}_n, \mathcal{T}_{\#_n})$, determine $x \in \mathcal{U}$. Since $\Pi^{\sigma}_{n \in \mathbb{N}}(\mathcal{V}_n, \mathcal{T}_{\#_n})$ is star-finite, following paragraph A.3.18 (def. (**d**)) we can define the subfann W_x of $V_{\Pi,\sigma}$. By $\mathbf{BT}^*$ (see 3.1.2), $\mathcal{U}$ is inductively open, therefore the bar $B^x_{\mathcal{U}} = \{b \in V_{\Pi,\sigma} \mid b \# x_{\lg(b)} \vee [b] \subseteq \mathcal{U}\}$ is inductive. By $\mathbf{HB}_{\propto}$ (prp. 3.2.1) $B^x_{\mathcal{U}}$ contains a finite subbar D on W_x, therefore we can find $N \in \mathbb{N}$ with ${}^N W_x \subseteq D_{\preceq}$.

Determine $a = a_0, \dots, a_N \in V_{\Pi,\sigma}$ such that $x \prec_\Pi a$. Then $a \in {}^N W_x$ and $[\![a]\!] \subseteq \mathcal{U}$. For $b = b_0, \dots, b_N \in V_{\Pi,\sigma}$, by the nature of D, if $b \approx_\Pi a$, then $[\![b]\!] \subseteq \mathcal{U}$. This implies that $\bigcap_{i \leq N} \pi_i^{-1}([\![C_i]\!]) \subseteq \mathcal{U}$, where $C_i = \{b \in {}^N V_i \mid b \approx_i a_i\}$, for $i \leq N$. However, $[\![C_i]\!]$ is a neighborhood of $x_{[i]}$ in $(\mathcal{V}_i, \mathcal{T}_{\#_i})$ for each $i \leq N$, which shows that $\mathcal{U}$ is open in the Tychonoff topology. (END OF PROOF)

A.3.14 Proof of lemma 3.5.3 We prove the technical lemma which is needed for theorem 3.5.3 (entailing a BISH version of Tychonoff's theorem).

LEMMA: (Notations as in 3.5.1) Let $a \in V_0, b \in V_1$ and let G, H be genetic bars on $(V_0)_a, (V_1)_b$ respectively. Then $G \overset{\sigma}{\times} H$ is an inductive bar on $(V_0)_a \overset{\sigma}{\times} (V_1)_b$.

COROLLARY: For $i \leq n$ let B_i be an inductive bar on V_i. Then $\Pi^\sigma_{i \leq n} B_i$ is an inductive bar on $V^{(n)}_{\Pi,\sigma}$ and $\overline{\Pi^\sigma_{i \leq n} B_i}$ is an inductive bar on $V_{\Pi,\sigma}$.

PROOF: By (double) genetic induction. For notational simplicity put $V_0 = V$ and $V_1 = W$. First let P_W be the following property of genetic bars B on basic subspraids W_b of W: 'for all $a \in V$: $\{\bigcirc_a\} \overset{\sigma}{\times} B$ is an inductive bar on $V_a \overset{\sigma}{\times} W_b$'. We show by genetic induction that all genetic bars B on basic subspraids W_b have property P_W. Let $a \in V, b \in W$.

- $\mathbf{G}_\bigcirc$ If $B = \{\bigcirc_b\}$, then $\{\bigcirc_a\} \overset{\sigma}{\times} B = \{(\bigcirc_a, \bigcirc_b)\}$ and we are done trivially.

- $\mathbf{G}_\propto$ Else $B = \bigcup_{b' \in \propto(b)} B_{b'}$ where $B_{b'}$ has property P_W for all $b' \in \propto(b)$. Then $\{a'\} \overset{\sigma}{\times} B_{b'}$ is an inductive bar on $V_{a'} \overset{\sigma}{\times} W_{b'}$ for all $a' \in \propto(a)$, which implies that $D = \bigcup_{a' \in \propto(a), b' \in \propto(b)} \{a'\} \overset{\sigma}{\times} B_{b'}$ is an inductive bar on $V_a \overset{\sigma}{\times} W_b$. Clearly $\{\bigcirc_a\} \overset{\sigma}{\times} B$ descends from D and so is inductive also.

By symmetry, for all genetic bars B on basic subspraids V_a of V we also obtain P_V: 'for all $b \in W$: $B \overset{\sigma}{\times} \{\bigcirc_b\}$ is an inductive bar on $V_a \overset{\sigma}{\times} W_b$'.
Next we show that all genetic bars B on basic subspraids V_a of V have property Q: 'for all $b \in W$ and every genetic bar C on W_b: $B \overset{\sigma}{\times} C$ is an inductive bar on $V_a \overset{\sigma}{\times} W_b$'. Let $a \in V, b \in W$.

- $\mathbf{G}_\bigcirc$ If $B = \{\bigcirc_a\}$, then we are done since every genetic bar on W_b has property P_W.

- $\mathbf{G}_\propto$ Else $B = \bigcup_{a' \in \propto(a)} B_{a'}$ where $B_{a'}$ has property Q for all $a' \in \propto(a)$. Let C be a genetic bar on W_b, we proceed by genetic induction on C.

  - $\mathbf{G}_\bigcirc$ If $C = \{\bigcirc_b\}$, then we are done since every genetic bar on V_a has property P_V.

G$_{\propto}$ Else $C = \bigcup_{b' \in \propto(b)} C_{b'}$ where by induction $B_{a'} \overset{\sigma}{\times} C_{b'}$ is an inductive bar on $V_{a'} \overset{\sigma}{\times} W_{b'}$ for all $a' \in \propto(a), b' \in \propto(b)$. Therefore we see that $D = \bigcup_{a' \in \propto(a) b' \in \propto(b)} \{a'\} \overset{\sigma}{\times} B_{b'}$ is an inductive bar on $V_a \overset{\sigma}{\times} W_b$. Clearly $B \overset{\sigma}{\times} C$ descends from D and so is inductive also.

This proves the lemma. The first part of the corollary follows by induction. For the second part we show by genetic induction that for every genetic bar B on a basic subspraid $(V_{\Pi,\sigma}^{(n)})_a$ of $V_{\Pi,\sigma}^{(n)}$ we have: $\overline{\{a\}} \blacktriangleleft_{\propto} \overline{B}$ in $V_{\Pi,\sigma}$. Let $a \in V_{\Pi,\sigma}^{(n)}$.

G$_{\bigcirc}$ If $B = \{\bigcirc_a\}$, then trivially $\overline{\{a\}} \blacktriangleleft_{\propto} \overline{B}$.

G$_{\propto}$ Else $B = \bigcup_{b \in \propto(a)} B_b$, where by induction $\overline{\{b\}} \blacktriangleleft_{\propto} \overline{B_b}$ for all $b \in \propto(a)$. We have $\overline{\{a\}} \blacktriangleleft_{\propto} \bigcup_{b \in \propto(a)} \overline{\{b\}}$ so $\overline{\{a\}} \blacktriangleleft_{\propto} \bigcup_{b \in \propto(a)} \overline{B_b} = \overline{B}$ and we are done.

Finally, $\overline{\{\bigcirc_{\Pi}^{(n)}\}}$ equals $\overline{{}^{n}V_{\Pi,\sigma}}$ which is an inductive bar on $V_{\Pi,\sigma}$ by lemma 3.3.1. So any inductive bar on $\overline{\{\bigcirc_{\Pi}^{(n)}\}}$ is an inductive bar on $V_{\Pi,\sigma}$. (END OF PROOF)

A.3.15 Defining various concepts of locatedness

(Partly repeating 4.0.1:) What are the drawbacks of the concept 'located in'? First of all, the notion is not transitive, which is unpractical when working with extensions and subspaces of (X, d). Second, even for a closed located $A \subset X$, the notion gives little handhold for $x \in X$ to find $a \in A$ such that $x \# a$ implies $x \# A$, which is an important prerequisite for many constructions involving A. Thirdly, as mentioned, the notion is non-topological and this means we cannot use it easily in the context of topology.

In [Waa1996] several alternatives are given in BISH, of which 'strongly halflocated in' (transitive) seems the most fruitful.[17] It gives results such as in the BISH-proof of the Dugundji extension theorem in [Waa1996]. Another result is that every complete metric space can be isometrically embedded in a normed linear extension such that it becomes strongly halflocated in this extension – and where we know of no general proof that it is located.

'Topologically strongly halflocated' in INT is equivalent on complete metric spaces to a topological locatedness property called 'strongly sublocated in'. This notion can also be defined for the apartness topology of general natural spaces, and seems to us important. Our definition of 'located in' is easily seen to be equivalent to the traditional definition, and in this form it opens the door for adaptations.

[17] By transitive we mean: if (B, d) is (strongly) halflocated in (A, d) which is (strongly) halflocated in (X, d) then (B, d) is (strongly) halflocated in (X, d).

DEFINITION: Let (A,d) be a subspace of (X,d), a metric space. Then (A,d) is (i) *located*, (ii) *halflocated*, (iii) *sublocated* in (X,d) iff: (A,d) is inhabited and

(i) $\forall D\in\mathbb{R}_{>1}\forall x\in X\forall m\in\mathbb{Z}\,[\,\exists a\in A\,[d(x,a)<D^{m+1}]\vee\forall a\in A\,[d(x,a)>D^m]]$

(ii) $\exists D\in\mathbb{R}_{>1}\forall x\in X\forall m\in\mathbb{Z}\,[\,\exists a\in A\,[d(x,a)<D^{m+1}]\vee\forall a\in A\,[d(x,a)>D^m]]$

(iii) $\forall x\in X\forall m\in\mathbb{Z}\,[\,\exists a\in A\,[d(x,a)<2^m]\vee\exists n\in\mathbb{N}\;\forall a\in A\,[d(x,a)>2^{-n}]]$

If $D\in\mathbb{R}_{>1}$ realizes (ii), we say that (A,d) is halflocated in (X,d) *with parameter D*. We now strengthen the definition:

(A,d) is (i)* *strongly located*, (ii)* *strongly halflocated*, (iii)* *strongly sublocated* in (X,d) iff:

(i)* $\forall D\in\mathbb{R}_{>1}\forall x\in X\exists y\in A\;\forall a\in A\,[\,d(x,y)\leq D\cdot d(x,a)\,]$

(ii)* $\exists D\in\mathbb{R}_{\geq 1}\forall x\in X\exists y\in A\;\forall a\in A\,[\,d(x,y)\leq D\cdot d(x,a)\,]$

(iii)* $\forall x\in X\exists y\in A\,[\,x\#y\rightarrow\exists n\in\mathbb{N}\;\forall a\in A\,[d(x,a)>2^{-n}]]$

If $D\in\mathbb{R}_{>1}$ realizes (ii), then (A,d) is strongly halflocated in (X,d) *with parameter D*. If (ii) is realized by $D=1$, then (A,d) is *best approximable* in (X,d).

Now let $(\mathcal{V},\mathcal{T}_\#)$ be a natural space derived from $(V,\#,\preceq)$, and let $(\mathcal{W},\mathcal{T}_\#)$ be a subspace. We say that $(\mathcal{W},\mathcal{T}_\#)$ is (i)* *#-sublocated*, (ii)* *strongly #-sublocated* in $(\mathcal{V},\mathcal{T}_\#)$ iff:

(i)* $\forall x\in\mathcal{V}\forall\,\mathcal{U}\in\mathcal{T}_\#,x\in\mathcal{U}\,[\,\exists y\in\mathcal{W}[y\in\mathcal{U}]\vee\forall y\in\mathcal{W}[x\#y]]$.

(ii)* $\forall x\in\mathcal{V}\exists y\in\mathcal{W}\,[\,x\#y\rightarrow\forall z\in\mathcal{W}\,[x\#z]]$.

(END OF DEFINITION)

REMARK: The word 'strongly' for (i)*-(iv)* and (ii)* is justified. For $x\in X$, if y realizes (ii)* with parameter D, we can decide: $d(x,y)<D^{m+1}$ or $d(x,y)>D^m$, for $m\in\mathbb{Z}$. But $d(x,y)>D^m$ implies that for all $a\in A$: $d(x,a)>D^{m-1}$. This shows that if (A,d) is strongly halflocated in (X,d), with parameter D, then (A,d) is halflocated in (X,d) with parameter D^2. Then 'strongly located in' implies 'located in' since $\{D^2\,|\,D\in\mathbb{R}_{>1}\}=\mathbb{R}_{>1}$. The other implications can be obtained in a similar but easier fashion. Notice that if (A,d) is strongly (half,sub)located in (X,d), then (A,d) is closed in (X,d). For a more extensive treatment of these properties, see [Waa1996]. (END OF REMARK)

A.3.16 Proof of proposition 4.0.5

Our example of the eventually vanishing real sequences still needs to be proven non-metrizable. We repeat from 4.0.5:

PROPOSITION: $(\overrightarrow{\mathbb{R}}^{\omega}, \overrightarrow{\mathcal{T}}^{\omega}_{\#})$ is a natural space which is the non-metrizable direct limit of $(\mathbb{R}^n_*, \overrightarrow{\mathcal{T}}^{\omega}_{\#})_{n\in\mathbb{N}}$, where $(\mathbb{R}^n_*, \overrightarrow{\mathcal{T}}^{\omega}_{\#})$ is $\preceq$-isomorphic to the Euclidean space $(\mathbb{R}^n, \mathcal{T}_{\#_{\mathbb{R}}})$ for $n\in\mathbb{N}^+$. There is a continuous injective surjection (which does not have a continuous inverse) from $(\overrightarrow{\mathbb{R}}^{\omega}, \overrightarrow{\mathcal{T}}^{\omega}_{\#})$ to $(\overrightarrow{\mathbb{R}}^{\omega}, \mathcal{T}_{\#_{\mathbb{R}^{\mathbb{N}}}})$ as subspace of $(\mathbb{R}^{\mathbb{N}}, \mathcal{T}_{\#_{\mathbb{R}^{\mathbb{N}}}})$.

PROOF: That $(\overrightarrow{\mathbb{R}}^{\omega}, \overrightarrow{\mathcal{T}}^{\omega}_{\#})$ is non-metrizable is seen thus: consider a metric d on $(\overrightarrow{\mathbb{R}}^{\omega}, \overrightarrow{\mathcal{T}}^{\omega}_{\#})$, then d is also a metric on the respective $(\mathbb{R}^n, \mathcal{T}_{\#_{\mathbb{R}}})$ for $n\in\mathbb{N}^+$ which respects the canonical inclusion relation $i_n : \mathbb{R}^n \to \mathbb{R}^{n+1}$. For $n\in\mathbb{N}^+$ let $0_{\mathbb{R}^n}$ be the origin in $\mathbb{R}^n$ and let $B^n_d(0_{\mathbb{R}^n}, 2^{-n})$ be the open d-sphere around this origin with radius 2^{-n}. We can construct a series of sets $(U_n)_{n\in\mathbb{N}^+}$ where each U_n is a d-open neighborhood in $\mathbb{R}^n$ of $0_{\mathbb{R}^n}$ and $i_n(U_n)\subset U_{n+1}$, and where in addition $B^n_d(0_{\mathbb{R}^n}, 2^{-n})$ contains a point which is not contained in U_n.

Remember that for $x\in\overrightarrow{\mathbb{R}}^{\omega}$ the m-th basic dot x_m (if not equal to the maximal dot) is a finite sequence of closed rational intervals $([a_j, b_j])_{j\leq s}$ for some $s\in\mathbb{N}$ and $a_j < b_j$ for all $j\leq s$. We identify x_m with the cartesian product set $\Pi(x_m)=\Pi_{j\leq s}[a_j, b_j]$ in the Euclidean space $\mathbb{R}^{s+1}$. Now it is not difficult to see that the subset $\{y \in \overrightarrow{\mathbb{R}}^{\omega} \mid \exists x\equiv y[\,\forall n\in\mathbb{N}[\Pi(x_n)\subset U_n]\,]\}$ is open in $\overrightarrow{\mathcal{T}}^{\omega}_{\#}$, but cannot be open in the metric topology generated by d.

Further, the identity on $\overrightarrow{\mathbb{R}}^{\omega}$ is a continuous injective surjection (which does not have a continuous inverse) from $(\overrightarrow{\mathbb{R}}^{\omega}, \overrightarrow{\mathcal{T}}^{\omega}_{\#})$ to $(\overrightarrow{\mathbb{R}}^{\omega}, \mathcal{T}_{\#_{\mathbb{R}^{\mathbb{N}}}})$ as subspace of $(\mathbb{R}^{\mathbb{N}}, \mathcal{T}_{\#_{\mathbb{R}^{\mathbb{N}}}})$. One sees this by considering that on $\mathbb{R}^{\mathbb{N}}$ the apartness topology $\mathcal{T}_{\#_{\mathbb{R}^{\mathbb{N}}}}$ coincides with the metric $d_{\mathbb{R}^{\mathbb{N}}}$-topology.

Finally, that $(\mathbb{R}^n_*, \overrightarrow{\mathcal{T}}^{\omega}_{\#})$ is $\preceq$-isomorphic to the Euclidean space $(\mathbb{R}^n, \mathcal{T}_{\#_{\mathbb{R}}})$ for $n\in\mathbb{N}^+$ is left to the reader as an exercise. (END OF PROOF)

A.3.17 Proof of corollary 4.0.8

For didactical reasons we prove the metrizability of $\propto$-fans first (the corollary of theorem 4.0.8), and metrizability of star-finite $\propto$-spreads (theorem 4.0.8 itself) in the next paragraph. The proof of the theorem for star-finite $\propto$-spreads employs the same main strategy, but is lengthy and involves some hard work on details, which tends to obscure this strategy.

The following two paragraphs contain the longest proof in the monograph.

In order to prove that every star-finitary space is metrizable, we will need several lemmas, to which we assign uppercase letters **A**, **B**, ... In a way we follow the classical strategy used by Urysohn (see [Ury1925b]) to show that (classically) a normal space $(X, \mathcal{T})$ with a countable base is metrizable. Urysohn embeds such an $(X, \mathcal{T})$ in the Hilbert cube, and this is what we do also for a star-finite $\propto$-spread, although our construction of such an embedding is different.[18]

For this construction we use the ternary real numbers $\mathbb{R}_{\text{ter}}$, and specifically $[0,1]_{\text{ter}}$. We refer the reader to the examples' section A.2.2 for the relevant definitions.

We rearrange the ingredients and the route followed in [Waa1996], partly in order to avoid the use of $\mathbf{AC}_{10}$.[19] This calls for a short description of our route beforehand. It is possible to prove metrization for $\propto$-fans, using the Urysohn metrization lemma (**A**), a splitting lemma (**B**), and the Urysohn function lemma (**C**) below.[20] To generalize this to a star-finite $\propto$-spread $(\mathcal{V}, \mathcal{T}_{\#})$, we associate to points x in $\mathcal{V}$ a subfan W_x of $(\mathcal{V}, \mathcal{T}_{\#})$ which almost acts as a neighborhood of x in $(\mathcal{V}, \mathcal{T}_{\#})$. Using an elaborate adaptation of lemma (**C**) we can again apply the Urysohn metrization lemma to conclude metrizability of $(\mathcal{V}, \mathcal{T}_{\#})$. All in all this will take up numerous pages.

In this and the following paragraph we need some definitions, to which we assign lowercase letters **a**, **b**, etc.

DEFINITION: (**a**)
Let $(\mathcal{V}, \mathcal{T}_{\#})$ be a spraid derived from $(V, \#, \preceq)$. Let $A \subseteq V$, then we write ${}^nA$ for $\{a \in A \mid \lg(a) = n\}$, for $n \in \mathbb{N}$, and $A_{\preceq}$ for $\{b \in V \mid \exists a \in A[b \preceq a]\}$. Also remember that we write $\approx$ for the touch-relation on basic dots which is the complement of the pre-apartness relation $\#$.

Now let $x \in \mathcal{V}$. We say that x is a *successor point* iff $\lg(x_n) = n$ for all $n \in \mathbb{N}$ (which implies $x_0 = \bigcirc$ and $x_{n+1} \propto x_n$ for $n \in \mathbb{N}$). One easily sees that any $y \in \mathcal{V}$ contains a unique subsequence $y^{\propto} \equiv y$ such that $y^{\propto}$ is a successor point. This means that without loss of generality we can conveniently concentrate on successor points.

[18] For one thing, it is a real construction. The key idea can also be seen as an onion strategy,but different from the classical one.

[19] We comment on the relation between the intuitionistic proof in [Waa1996] and the proof here, in the comments' section, paragraph A.5.7.

[20] The term 'Urysohn's lemma' is usually reserved for the related classical result that a space is 'normal' iff two disjoint closed subsets can be separated by a continuous function to $[0,1]$. See our lemma (**C**) and its generalization (**E**) which we call 'Urysohn function lemma'.

We wish to expand $(\mathcal{V},\mathcal{T}_{\#})$ with a single isolated point, which w.l.o.g. can be taken to be $\underline{\bullet} = \bullet,\bullet,\ldots$. Strictly speaking we put $V^{\bullet} \underset{D}{=} V \cup \{\bullet\}^{*}$ where $\{\bullet\}^{*} = \{\overline{\bullet}(n) \mid n\in\mathbb{N}, n\geq 1\}$ is the set of all finite sequences of the symbol $\bullet$, and specify that $\bigcirc \succeq \overline{\bullet}(n) \succeq \overline{\bullet}(n+1)$ and $\overline{\bullet}(n)\#a$ for all $\bigcirc \neq a\in V$, $n\geq 1$. The resulting spraid is denoted $(\mathcal{V}^{\bullet},\mathcal{T}_{\#})$ or simply $\mathcal{V}^{\bullet}$. (END OF DEFINITION)

LEMMA: (**A**) (Urysohn metrization lemma)
Let $(\mathcal{V},\mathcal{T}_{\#})$ be a spraid derived from $(V,\#,\preceq)$, where we write $\approx$ for the complement of the pre-apartness relation $\#$. Suppose that:

(i) For all $n\in\mathbb{N}$, for all $a\#b\in{}^{n}V^{\bullet}$ there is a given morphism $f_{a,b}$ from $\mathcal{V}^{\bullet}$ to $[0,1]$, such that $f_{a,b}|_{V^{\bullet}_{a}} \equiv_{\mathbb{R}} 0$ and $f_{a,b}|_{V^{\bullet}_{b}} \equiv_{\mathbb{R}} 1$.

(ii) For all $n\in\mathbb{N}$ and $a\#b\in{}^{n}V^{\bullet}$: if $c\in{}^{n}V^{\bullet}$ with $a\#c\#b$ then $f_{a,b}|_{V^{\bullet}_{b}} \subseteq [\frac{1}{3},\frac{2}{3}]$

(iii) For any $\#$-open $\mathcal{U}\subseteq\mathcal{V}$ and successor point $x\in\mathcal{U}$ there is an $n\in\mathbb{N}$ such that for all $a\in{}^{n}V$ we have: $a\approx x_n$ implies $[a]\subseteq\mathcal{U}$.

Then $(\mathcal{V},\mathcal{T}_{\#})$ is metrizable.

PROOF: There is of course a canonical morphism f from $\mathcal{V}^{\bullet}$ to $[0,1]$ such that $f(x)\equiv_{\mathbb{R}}0$ for all $x\in\mathcal{V}^{\bullet}$. So by the assumption (i), we can define for all $n\in\mathbb{N}$ and all $a,b\in{}^{n}V^{\bullet}$ a morphism $f_{a,b}$ such that $a\approx b$ implies $f_{a,b}(x)\equiv_{\mathbb{R}}0$ for all $x\in\mathcal{V}^{\bullet}$, and $a\#b$ implies $f_{a,b}|_{V^{\bullet}_{a}} \equiv_{\mathbb{R}} 0$ and $f_{a,b}|_{V^{\bullet}_{b}} \equiv_{\mathbb{R}} 1$.

Now let h be an enumeration of $\bigcup_{n\in\mathbb{N}}{}^{n}V^{\bullet}\times{}^{n}V^{\bullet}$. Define a metric d on $\mathcal{V}^{\bullet}$ by putting $d(x,y)=\sum_{m\in\mathbb{N}}2^{-m}\cdot|f_{h(m)}(x)-f_{h(m)}(y)|$. Then we see that $d(x,y)>0$ iff $x\#y$, and that d is a metric on $\mathcal{V}^{\bullet}$ (this shows weak metrizability).

To show that d metrizes $(\mathcal{V},\mathcal{T}_{\#})$, let $\mathcal{U}$ be $\#$-open in$(\mathcal{V},\mathcal{T}_{\#})$. We show that $\mathcal{U}$ is d-open as well. For this let $x\in\mathcal{U}$ be a successor point. By assumption (iii) there is an $n\in\mathbb{N}$ such that for all $a\in{}^{n}V$ we have: $a\approx x_n$ implies $[a]\subseteq\mathcal{U}$. Now suppose we have a successor point $y\in\mathcal{V}$ such that $y_n\#x_n$. Determine $m\in\mathbb{N}$ such that $h(m)=(\overline{\bullet}(n),x_n)$. Clearly $\overline{\bullet}(n)\#y_n\#x_n\#\overline{\bullet}(n)$, so by assumption (ii) we see that $f_{\overline{\bullet}(n),x_n}(y)\in[\frac{1}{3},\frac{2}{3}]$, whereas $f_{\overline{\bullet}(n),x_n}(x)\equiv_{\mathbb{R}}1$.
This implies that $d(x,y)>2^{-m-1}\cdot\frac{1}{3}$ and so in turn that $B(x,2^{-m-1}\cdot\frac{1}{3})\subseteq\mathcal{U}$. Since $x\in\mathcal{U}$ is arbitrary, this shows that $\mathcal{U}$ is d-open as well. (In fact d metrizes $\mathcal{V}^{\bullet}$, but this is unimportant). (END OF PROOF)

We turn to our splitting lemma. Remember that for a spraid $(\mathcal{V},\mathcal{T}_{\#})$ derived from $(V,\#,\preceq)$, and subsets A,B of V we write $A\#B$ iff $a\#b$ for all $a\in A, b\in B$. We write $A\approx B$ iff $a\approx b$ for some $a\in A, b\in B$. We shortly write $a\#B$, $a\approx B$ for $\{a\}\#B$, $\{a\}\approx B$ respectively.

LEMMA: (**B**) (splitting lemma)
Let $(\mathcal{W}, \mathcal{T}_{\#})$ be a $\propto$-fan derived from $(W, \#, \preceq)$. Suppose A, B are finite subsets of W such that $A \# B$. Then there is an $N \in \mathbb{N}$ such that for all $c, d \in {}^N W$ we have: $(c \approx A \wedge d \approx B)$ implies $c \# d$ (and $(c \approx {}^N A_{\preceq} \wedge d \approx {}^N B_{\preceq})$ implies $c \# d$).

PROOF: By proposition 3.4.1(ii), $C = \{c \in W \mid c \# A \vee c \# B\}$ is an inductive bar on $\mathcal{W}$. By $\mathbf{HB}_{\propto}$ (crl. 3.2.1) C contains a finite bar C'. Let $A' = \{c \in C' \mid c \approx A\}$, then A' is finite and $A' \# B$. So by repeating our argument we find a finite bar C'' on $\mathcal{W}$ such that for all $c \in C''$ we have $c \# A'$ or $c \# B$. Put $N = \max(\{\lg(c) \mid c \in C'' \cup A \cup B\})$. Then for all $c, d \in {}^N W$ we have: $(c \approx A \wedge d \approx B)$ implies $c \# d$ (and $(c \approx {}^N A_{\preceq} \wedge d \approx {}^N B_{\preceq})$ implies $c \# d$). (END OF PROOF)

The splitting lemma tells us that for a $\propto$-fan $(\mathcal{W}, \mathcal{T}_{\#})$, finite apart $A \# B \subset W$ lead (for big enough $N \in \mathbb{N}$) to a partition of ${}^N W$ in three sets C, D, E where $A \approx C$, $A \# D \# B$, $E \approx B$ and moreover $C \# E$. We will use such partitions obtained by lemma (**B**) to construct morphisms from $(\mathcal{W}, \mathcal{T}_{\#})$ to $\sigma_{3,\text{ter}}$. In these constructions, to such C we assign a 0, to such D a 1 and to such E a 2.

For this, remember our definition in A.2.2 of $\sigma_3 = (\{0,1,2\}^*, \#_{\omega}, \preceq_{\omega})$ and the surjective morphism $f_{\text{evl},3}$ from σ_3 to $[0,1]_{\text{ter}}$. Pulling back $\#_{\mathbb{R}}$ using $f_{\text{evl},3}$ we saw that $f_{\text{evl},3}$ is a $\preceq$-isomorphism from $\sigma_{3,\text{ter}} = (\{0,1,2\}^*, \#_{\mathbb{R}}, \preceq_{\omega})$ to $[0,1]_{\text{ter}}$.

DEFINITION: (**b**)
We define a lexicographical ordering $<_{\text{lex}}$ on $\mathbb{N}^*$ putting, for $a = a_0, \ldots a_{n-1}$ and $b = b_0, \ldots, b_{m-1}$ in $\mathbb{N}^*$:

$a <_{\text{lex}} b$ iff $(b \prec a \vee \exists i < n, m[a_0, \ldots, a_{i-1} = b_0, \ldots, b_{i-1} \wedge a_i < b_i])$.

By slight abuse of notation we write ${}^n\sigma_3$ for $\{a \in \{0,1,2\}^* \mid \lg(a) = n\}$. For each $n \in \mathbb{N}$, $<_{\text{lex}}$ induces a finite linear ordering on ${}^n\sigma_3$. For $a \in {}^n\sigma_3, a \neq \underline{\overline{0}}(n)$ we write Pre(a) for the immediate predecessor of a in this ordering. For $a \in {}^n\sigma_3, a \neq \underline{\overline{2}}(n)$ we write Suc(a) for the immediate successor of a in this ordering. Additionally we put Pre$(\underline{\overline{0}}(n)) \underset{D}{=} -1$ and Suc$(\underline{\overline{2}}(n)) \underset{D}{=} 3$. (END OF DEFINITION)

LEMMA: (**C**) (Urysohn function lemma for fans)
Let $(\mathcal{W}, \mathcal{T}_{\#})$ be a $\propto$-fan derived from $(W, \#, \preceq)$. Let $a \# b \in {}^m W$ for certain $m \in \mathbb{N}$. Then there is a canonical morphism $f_{a,b}$ from $\mathcal{W}$ to $[0,1]$ such that:

(i) $f_{a,b}|_{W_a} \equiv_{\mathbb{R}} 0$ and $f_{a,b}|_{W_b} \equiv_{\mathbb{R}} 1$.

(ii) If $c \in {}^m W$ with $a \# c \# b$ then $f_{a,b}|_{W_c} \subseteq [\frac{1}{3}, \frac{2}{3}]$

PROOF: We inductively define, for all $i \in \{0,1,2\}^* \cup \{-1,3\}$, a subset $W_i \subset W$ such that for all $n \in \mathbb{N}$:

(∗) For $i,j \in {}^n\sigma_3 \cup \{-1,3\}$ the set W_i is decidable. Moreover $W_i \# W_j$ whenever $j \notin \{\mathrm{Pre}(i), i, \mathrm{Suc}(i)\}$, and if $i \neq j$ are in ${}^n\sigma_3$ then $W_i \cap W_j = \emptyset$.

(∗∗) For $i \in \{0,1,2\}^*$ and $s \in \{0,1,2\}$ we have $W_{i \star s} \subset W_i$.

(∗∗∗) There is $t \in \mathbb{N}$ such that all $c \in {}^tW$ are in some W_i where $i \in {}^n\sigma_3$.

Basis: for $n=0$, put $W_{\bigcirc} = W$, $W_{-1} = W_a$ and $W_3 = W_b$, then $W_{-1} \# W_3$, and (∗∗) and (∗∗∗) are trivially fulfilled so we are done.

Induction: suppose for $n \in \mathbb{N}$ and $i \in {}^n\sigma_3$ the subsets W_i have been defined such that (∗), (∗∗), and (∗∗∗) above hold.
Let $i \in {}^n\sigma_3$. Determine the least $t \in \mathbb{N}$ such that all $c \in {}^tW$ are in some W_j for $j \in {}^n\sigma_3$. Now put $A = W_{\mathrm{Pre}(i)} \cap {}^tW$ and $B = W_{\mathrm{Suc}(i)} \cap {}^tW$. Then by (∗) we see that $A \# B \subset W$ are finite subsets of W, so by lemma (**B**) we can find the smallest $N \geq t+1$ such that for all $c,d \in {}^NW$ we have: $(c \approx A \wedge d \approx B)$ implies $c \# d$.
Let $C = \{c \in {}^NW \mid c \approx A\}$, $D = \{c \in {}^NW \mid A \# c \# B\}$, and $E = \{c \in {}^NW \mid c \approx B\}$. Then $A \# D \# B$ and $C \# E$. Put $W_{i\star 0} = C_{\preceq} \cap W_i$, $W_{i\star 1} = D_{\preceq} \cap W_i$, and $W_{i\star 2} = E_{\preceq} \cap W_i$.

This defines W_i for all $i \in {}^{n+1}\sigma_3$. It is straightforward to see that (∗) and (∗∗) hold for $n+1$. To see that (∗∗∗) holds as well, it suffices to consider that ${}^n\sigma_3$ is finite, and that therefore in our procedure above there was $j \in {}^n\sigma_3$ with (copying notations) a maximal $N_j \geq t+1$ compared to other $i \in {}^n\sigma_3$. This means that all $c \in {}^{N_j}W$ are in some W_i for $i \in {}^{n+1}\sigma_3$.

We define a $\preceq$-morphism $h_{a,b}$ from $\mathcal{W}$ to $\sigma_{3,\mathrm{ter}}$ by specifying $h_{a,b}$ on W. Let $c \in W$, then there is a maximal $s \in \mathbb{N}, s \leq \lg(c)$ such that c is in W_i for a unique $i \in {}^s\sigma_3$. We now simply put $h_{a,b}(c) = i$.

claim $h_{a,b}$ is a $\preceq$-morphism from $\mathcal{W}$ to $\sigma_{3,\mathrm{ter}}$.

proof It follows from (∗), (∗∗) and (∗∗∗) above that for $c \preceq d \in W$ we have $h_{a,b}(c) \preceq h_{a,b}(d)$. Let $x \in \mathcal{W}$ be a successor point, we need to show that $h_{a,b}(x)$ is a point in σ_3. This follows easily however from (∗∗∗) above, since we see that for all $n \in \mathbb{N}$ there is a $t \in \mathbb{N}$ such that $h_{a,b}(x_t) \in {}^n\sigma_3$. Finally, suppose $y \in \mathcal{W}$ is a successor point such that $h_{a,b}(y) \#_{\mathbb{R}} h_{a,b}(x)$, then we must show $y \# x$. Since $h_{a,b}(y) \#_{\mathbb{R}} h_{a,b}(x)$, we can determine $n \in \mathbb{N}$ and $i,j \in {}^n\sigma_3$ such that $i \#_{\mathbb{R}} j$ and $h_{a,b}(y) \prec i$ and $h_{a,b}(x) \prec j$. However, $i \#_{\mathbb{R}} j$ equals $j \notin \{\mathrm{Pre}(i), i, \mathrm{Suc}(i)\}$. There is $s \in \mathbb{N}$ such that $y_s \in W_i$ and $x_s \in W_j$, where moreover $W_i \# W_j$ since $j \notin \{\mathrm{Pre}(i), i, \mathrm{Suc}(i)\}$. So we see that $y_s \# x_s$ and so $y \# x$. (end of claim-proof)

We can now define $f_{a,b} \underset{D}{=} f_{\mathrm{evl},3} \circ h_{a,b}$ (which is canonical since $h_{a,b}$ and $f_{\mathrm{evl},3}$ are constructed canonically). Clearly $f_{a,b}|_{W_a} \equiv_{\mathbb{R}} 0$ and $f_{a,b}|_{W_b} \equiv_{\mathbb{R}} 1$ (proving (i) of the lemma). So the only thing left to prove is that if $c \in {}^m W$ with $a \# c \# b$ then $f_{a,b}|_{W_c} \subseteq [\frac{1}{3}, \frac{2}{3}]$. And this follows trivially from our construction, since there is $s \in \mathbb{N}$ such that $\{d \in W_c \mid \mathrm{lg}(d) \geq s\} \subset W_1$ and $f_{a,b}|_{W_1} \subseteq [\frac{1}{3}, \frac{2}{3}]$. (END OF PROOF)

PROPOSITION: Every $\propto$-fan is metrizable.

PROOF: Let $(\mathcal{W}, \mathcal{T}_\#)$ be a $\propto$-fan. Then trivially $\mathcal{W}^\bullet$ is also a $\propto$-fan. Therefore by lemma (**C**) above, we find:

(i) For all $n \in \mathbb{N}$, for all $a \# b \in {}^n W^\bullet$ there is a given morphism $f_{a,b}$ from $\mathcal{W}^\bullet$ to $[0,1]$, such that $f_{a,b}|_{W_a^\bullet} \equiv_{\mathbb{R}} 0$ and $f_{a,b}|_{W_b^\bullet} \equiv_{\mathbb{R}} 1$.

(ii) For all $n \in \mathbb{N}$ and $a \# b \in {}^n W^\bullet$: if $c \in {}^n W^\bullet$ with $a \# c \# b$ then $f_{a,b}|_{W_c^\bullet} \subseteq [\frac{1}{3}, \frac{2}{3}]$.

claim Let $\mathcal{U} \subseteq \mathcal{W}$ be $\#$-open and let $x \in \mathcal{U}$ be a successor point. Then there is an $n \in \mathbb{N}$ such that for all $a \in {}^n W$ we have: $a \approx x_n$ implies $[a] \subseteq \mathcal{U}$.

proof By definition 3.4.1 of '$\propto$-spread', $\mathcal{U}$ is $\propto$-open, which means that $B^x_{\mathcal{U}} = \{b \in W \mid b \# x_{\mathrm{lg}(b)} \vee [b] \subseteq \mathcal{U}\}$ is an inductive bar on W. Now since $\mathcal{W}$ is a fan, by $\mathbf{HB}_\propto$ (crl. 3.2.1) we find a finite subbar $B' \subseteq B^x_{\mathcal{U}}$ on W. Let n be the maximum of $\{\mathrm{lg}(b) \mid b \in B'\}$. Let $a \in {}^n W$. We know by the properties of B' that $a \# x_n$ or $[a] \subseteq \mathcal{U}$. Therefore $a \approx x_n$ implies $[a] \subseteq \mathcal{U}$. (end of claim-proof)

By the claim, we see that $(\mathcal{W}, \mathcal{T}_\#)$ satisfies the conditions (i), (ii) and (iii) of the Urysohn lemma (**A**). Therefore $(\mathcal{W}, \mathcal{T}_\#)$ is metrizable. (END OF PROOF)

To prove corollary 4.0.8, we define 'one-point $\propto$-fanlike extension' (see also 3.4.2) to represent 'locally compact' spaces, just as $\propto$-fanlike spaces represent 'compact' spaces.[21]

DEFINITION: (**c**)
Let $(\mathcal{V}, \mathcal{T}_\#)$ be a spread, then $(\mathcal{V}, \mathcal{T}_\#)$ has a *one-point $\propto$-fanlike extension* iff there is a $\propto$-fan $(\mathcal{W}, \mathcal{T}_{\#_2})$ (derived from $(W, \#_2, \preceq_2)$) and a function f from $V^\bullet$ to W such that putting $a \#_1 b$ iff $f(a) \#_2 f(b)$, we have that f is an inductive isomorphism from $(\mathcal{V}^\bullet, \mathcal{T}_{\#_1})$ to $(\mathcal{W}, \mathcal{T}_{\#_2})$ where in addition $f(\underline{\bullet}) \#_2 f(x)$ for all $x \in \mathcal{V}$ and $(\mathcal{V}, \mathcal{T}_{\#_1})$ is identically automorphic to $(\mathcal{V}, \mathcal{T}_\#)$. More generally,

[21] The difference with the CLASS and BISH notions is that our analogons need not be locally metrically complete. This can be addressed satisfactorily, in the sense that local completeness turns out to be a natural-topological property (invariant under isomorphisms). But we will leave this for subsequent expositions on natural topology, hopefully written by others.

a natural space $(\mathcal{V},\mathcal{T}_{\#})$ is said to have a one-point $\propto$-fanlike extension iff $(\mathcal{V},\mathcal{T}_{\#})$ is $\propto$-isomorphic to a spread with a one-point $\propto$-fanlike extension. (END OF DEFINITION)

COROLLARY: (which is the same as corollary 4.0.8) Every $\propto$-fanlike space is metrizable ('every compact space is metrizable'), and every space with a one-point $\propto$-fanlike extension is metrizable ('every locally compact space is metrizable').

PROOF: We only need to prove that a spread $(\mathcal{V},\mathcal{T}_{\#})$ (derived from $(V, \#, \preceq)$) with a one-point $\propto$-fanlike extension is metrizable. For this, let $(\mathcal{W},\mathcal{T}_{\#_2})$ be a $\propto$-fan (derived from $(W, \#_2, \preceq_2)$) and f a function from $V^{\bullet}$ to W such that putting $a\,\#_1 b$ iff $f(a)\,\#_2\, f(b)$, we have that f is an inductive isomorphism from $(\mathcal{V}^{\bullet},\mathcal{T}_{\#_1})$ to $(\mathcal{W},\mathcal{T}_{\#_2})$ where in addition $f(\underline{\bullet})\,\#_2\, f(x)$ for all $x\in\mathcal{V}$.

By the above proposition $(\mathcal{W},\mathcal{T}_{\#_2})$ is metrizable. This means that $(\mathcal{V}^{\bullet},\mathcal{T}_{\#_1})$ is metrizable, say by metric d. Then the restriction of d to $\mathcal{V}$ metrizes $(\mathcal{V},\mathcal{T}_{\#})$. This follows trivially from the fact that $\mathcal{T}_{\#}\mid_{\mathcal{V}} = \mathcal{T}_{\#_1}\mid_{\mathcal{V}}$, which in turn follows trivially from the fact that $\underline{\bullet}\,\#_1 x$ for all $x\in\mathcal{V}$. (END OF PROOF)

A.3.18 Proof of theorem 4.0.8

The key to the generalization of our results in paragraph A.3.17 is the following simple observation. In a star-finite $(\mathcal{V},\mathcal{T}_{\#})$, for each x in $\mathcal{V}$ the equivalence class of x, that is $\{y\in\mathcal{V}\,|\,y\equiv x\}$ is contained in a subfan W_x of $(\mathcal{V},\mathcal{T}_{\#})$. The spine of this subfan W_x is formed by $\{a\in V\,|\,a\approx x_m\,|\,\lg(x_m)=\lg(a)\}$. But we need to do some extra work to turn this spine into a subfan. Specifically, if a is in the spine but all continuations $b\prec a$ are seen to be not in this spine (which is decidable since $(\mathcal{V},\mathcal{T}_{\#})$ is star-finite), we need to ensure that a still contains a point in W_x.

To avoid cumbersome repetition of the prevailing conditions, we state the following:

CONVENTION: From now on, without loss of generality, we assume $(\mathcal{V},\mathcal{T}_{\#})$ to be a star-finite $\propto$-spread, derived from $(V, \#, \preceq)$ where $v : \mathbb{N} \to V$ is an enumeration of $V=\{v_n\,|\,n\in\mathbb{N}\}$ such that for each $n,m\in\mathbb{N}$ we have that $v_n\prec v_m$ implies $n>m$. The touch-relation $\approx$ is the complement of $\#$ on $V\times V$. We concentrate on successor points (see def. (**a**) above). Also we write '$m=\mu s\in\mathbb{N}[\mathrm{P}(s)]$' as abbreviation for '$m$ is the smallest natural number for which $\mathrm{P}(m)$ holds'. (END OF CONVENTION)

DEFINITION: (**d**)
Let $x\in\mathcal{V}$ be a successor point. Using the enumeration v we define a subfan W_x of $(\mathcal{V},\mathcal{T}_\#)$ by inductively describing ${}^nW_x$ for each $n\in\mathbb{N}$. Of course ${}^0W_x \underset{D}{=} \{\bigcirc\}$. Now suppose that ${}^nW_x$ has been defined for given $n\in\mathbb{N}$, then put

$$ {}^{n+1}W_x \underset{D}{=} \{v_m\in{}^{n+1}V \mid v_m\approx x_{n+1}\vee \\ \exists a\in{}^nW_x\,\forall b\in\propto(a)[b\#x_{n+1}\wedge m=\mu s\in\mathbb{N}[v_s\prec a]]\}. $$

Then W_x is a subfan of $(\mathcal{V},\mathcal{T}_\#)$ such that $\{y\in\mathcal{V}\,|\,y\equiv x\}\subseteq W_x$. For $n\in\mathbb{N}$, we are mostly interested in the subset ${}^nW_x^{\approx}\underset{D}{=}\{a\in{}^nW_x\,|\,a\approx x_n\}$ of ${}^nW_x$. Finally, let $y\in\mathcal{V}$ be arbitrary, then y has as subsequence the successor point $y^{\propto}\equiv y$ (see def. (**a**) above), and we put $W_y=W_{y^\propto}$. (END OF DEFINITION)

Next we show that W_x acts almost like a neighborhood of x in $(\mathcal{V},\mathcal{T}_\#)$. We need this later on, as prerequisite (iii) of the Urysohn metrization lemma.

LEMMA: (**D**)
Let $\mathcal{U}$ be open in $(\mathcal{V},\mathcal{T}_\#)$, and let $x\in\mathcal{U}$. Then there is $N\in\mathbb{N}$ such that for all $a\in{}^NV$ we have: $a\approx x_N$ implies $[a]\subseteq\mathcal{U}$.

PROOF: By definition 3.4.1 of '$\propto$-spread', $\mathcal{U}$ is $\propto$-open, which means that $B^x_{\mathcal{U}}=\{b\in V\,|\,b\#x_{\lg(b)}\vee[b]\subseteq\mathcal{U}\}$ is an inductive bar on V. Now since W_x is a subfan and $B^x_{\mathcal{U}}$ is monotone, by $\mathbf{HB}_\propto$ (crl. 3.2.1) we find a finite subbar $B'\subseteq B^x_{\mathcal{U}}$ on W_x. Let N be the maximum of $\{\lg(b)\,|\,b\in B'\}$. Let $a\in{}^NV$. We know that if $a\notin W_x$ then $a\#x_N$. Else, if $a\in W_x$ we know by the properties of B' that $a\#x_N$ or $[a]\subseteq\mathcal{U}$. We combine this to conclude for all $a\in{}^NV$ that $a\approx x_N$ implies $[a]\subseteq\mathcal{U}$. (END OF PROOF)

We need a sequence $(W_{x,n})_{n\in\mathbb{N}}$ of very similar but slightly larger fans than W_x for our purposes, where $W_{x,n}\subseteq W_{x,n+1}$ for each $n\in\mathbb{N}$. For this we expand our touch relation $\approx$ inductively to equivalent touch-relations $(\overset{n}{\approx})_{n\in\mathbb{N}}$ ('equivalent' meaning that they induce the same apartness relation on points).

DEFINITION: (**e**)
Let $\approx'\subset V\times V$, and write $\#'$ for the complement of $\approx'$. We say that $\approx'$ is a *$\approx$-equivalent touch-relation* iff $\approx'$ is decidable and for all points x,y in $\mathcal{V}$, we have: $x\#y$ iff there is $n\in\mathbb{N}$ with $x_n\#'y_n$. We say that $\approx'$ is star-finite iff for each $a\in V$ the subset $\{b\in V\,|\,\lg(b)=\lg(a)\wedge b\approx'a\}$ is finite.

We inductively define decidable relations $(\overset{n}{\approx})_{n\in\mathbb{N}}$ which are $\approx$-equivalent for $n\geq 1$ as follows. Let $\overset{0}{\approx}=\{(a,b)\,|\,a,b\in V\,|\,a\preceq b\vee b\preceq a\}$. Suppose for $n\in\mathbb{N}$ that $\overset{n}{\approx}$ has been defined. Let $a,b\in V$ with $m=\lg(a)\leq\lg(b)$. Then $a\overset{n+1}{\approx}b$ and

$b \overset{n+1}{\approx} a$ iff there is $c \in {}^mV$ such that $a \overset{n}{\approx} c$ and $c \approx b$. (The idea is that $a \overset{n}{\approx} b$ iff there is a $\approx$-trail of length n from a to b which does not use basic dots d with $\lg(d) < \lg(a) \leq \lg(b)$. Notice that $\overset{1}{\approx}$ equals $\approx$.).[22]

Let $a \in {}^mV$ for certain $m \in \mathbb{N}$, then we let $\mathrm{St}_n(a) = \{b \in {}^mV \mid a \overset{n}{\approx} b\}$, which is a finite set. For $\mathrm{St}_1(a)$ we also simply write $\mathrm{St}(a)$. For the complement of $\overset{n}{\approx}$ we write $\overset{n}{\not\approx}$ and for subsets A, B of V we write $A \overset{n}{\not\approx} B$ iff $a \overset{n}{\not\approx} b$ for all $a \in A, b \in B$. We write $A \overset{n}{\approx} B$ iff $a \overset{n}{\approx} b$ for some $a \in A, b \in B$. We shortly write $a \overset{n}{\not\approx} B$, $a \overset{n}{\approx} B$ for $\{a\} \overset{n}{\not\approx} B$, $\{a\} \overset{n}{\approx} B$ respectively.

For $x \in \mathcal{V}$ we define $W_{x,n}$ as the subfan of $(\mathcal{V}, \mathcal{T}_{\#})$ which we obtain by substituting $\overset{n}{\approx}$ for $\approx$ and $\overset{n}{\not\approx}$ for $\#$ in the definition (**d**) of W_x above.

Finally, for $n, m \in \mathbb{N}$ put ${}^nW^{\propto}_{x,m} \underset{D}{=} \{a \in {}^nW_{x,m} \mid a \overset{m}{\approx} x^{\propto}_n\}$, where $x^{\propto} \equiv x$ is the relevant successor point, see def. (**a**). (END OF DEFINITION)

To see that the definition of $W_{x,n}$ is valid, it suffices to check that $\overset{n}{\approx}$ is again star-finite (for $a \in V$, the set $\{b \in V \mid \lg(b) = \lg(a) \wedge b \overset{n}{\approx} a\}$ is finite).

Lemma (**B**) above can now be abbreviated thus: let $(\mathcal{W}, \mathcal{T}_{\#})$ be a $\propto$-fan derived from $(W, \#, \preceq)$. Suppose A, B are finite subsets of W such that $A \# B$. Then there is an $N \in \mathbb{N}$ such that ${}^NA_{\preceq} \overset{3}{\not\approx} {}^NB_{\preceq}$.[23]

We use lemma (**B**) to generalize the Urysohn function lemma (**C**) to $(\mathcal{V}, \mathcal{T}_{\#})$. This is an arduous task, which we try to make palatable by dividing the proof in two parts. In the first part we detail the construction (for $a \# b \in {}^MV$ and for $i \in \{0,1,2\}^* \cup \{-1,3\}$) of subsets $V_i \subset V$ such that in analogy to lemma (**C**) for $i, j \in {}^n\sigma_3 \cup \{-1,3\}$ the set V_i is decidable; if $j \notin \{\mathrm{Pre}(i), i, \mathrm{Suc}(i)\}$ then ${}^mV_i \# {}^mV_j$ for all $m \geq M$, and for $i, j \in {}^n\sigma_3$: if $i \neq j$ then $V_i \cap V_j = \emptyset$. Also for $i \in \{0,1,2\}^*$ and $s \in \{0,1,2\}$ we have $V_{i\star s} \subset V_i$.
In the second part of the proof these sets will help us to define a morphism $h_{a,b}$ to $\sigma_{3,\mathrm{ter}}$, which using $f_{\mathrm{evl},3}$ can be turned into the desired morphism $f_{a,b}$.

LEMMA: (**E**) (Urysohn function lemma, generalizing lemma (**C**))
Let the basic dots $a \# b$ be in ${}^MV$ for certain $M \in \mathbb{N}$. Then there is a canonical morphism $f_{a,b}$ from $\mathcal{V}$ to $[0,1]$ such that:

(i) $f_{a,b}|_{V_a} \equiv_{\mathbb{R}} 0$ and $f_{a,b}|_{V_b} \equiv_{\mathbb{R}} 1$.

(ii) If $c \in {}^MV$ with $a \# c \# b$ then $f_{a,b}|_{V_c} \subseteq [\frac{1}{3}, \frac{2}{3}]$

[22] The touch-relation $\overset{0}{\approx}$ could also be named $\approx_\omega$ since it corresponds to the 'naked' spread $(\mathcal{V}, \mathcal{T}_{\#_\omega})$ where we have stripped $\mathcal{V}$ of the apartness relation $\#$. Compare this to our discussion of $\mathbb{R}_{\#_\omega}$ derived from $(\sigma_{\mathbb{R}}^{\propto}, \#_\omega, \preceq)$ and Hawk-Eye in example A.2.0.

[23] This shows that $\overset{n}{\approx}$ is $\approx$-equivalent for $n \geq 1$.

PROOF:

<u>Part one</u>
We first put $V_{-1} = V_a$ and $V_3 = V_b$. Then $V_{-1} \# V_3$.
We then inductively define, for $n \in \mathbb{N}$ and $i \in {}^n\sigma_3$, subsets $V_i \subset V$ as well as a subset $\mathrm{Sec}_n \subset V$ such that (except for $i = \bigcirc_\omega \in {}^0\sigma_3$) a basic dot c is member of any of these sets iff $\lg(c) \geq M$, and in addition we have:

(∗) For $i \in {}^n\sigma_3$ the set V_i is decidable and $V_i = (V_i)_{\preceq}$.

(∗∗) For $i, j \in {}^n\sigma_3 \cup \{-1, 3\}$, if $j \notin \{\mathrm{Pre}(i), i, \mathrm{Suc}(i)\}$ then $V_i \# V_j$, and if $i \neq j$ are in ${}^n\sigma_3$ then $V_i \cap V_j = \emptyset$.

(∗∗∗) For $j \in {}^{n-1}\sigma_3$ and $s \in \{0, 1, 2\}$ we have $V_{j\star s} \subseteq V_j$.

When our construction is done, we find that for all $x \in \mathcal{V}$ and all $n \in \mathbb{N}$ there is $t \in \mathbb{N}$ and $i \in {}^n\sigma_3$ such that $x_t \in V_i$

Basis: for $n = 0$, the only member of ${}^n\sigma_3$ is $\bigcirc_\omega$. Put $V_{\bigcirc_\omega} = V$. Then $V_{-1} \# V_3$, and (∗), (∗∗) and (∗∗∗) are trivially fulfilled so we are done.

Induction: suppose for $n \in \mathbb{N}$ and all $i \in {}^n\sigma_3$ the subsets V_i with properties (∗), (∗∗) and (∗∗∗) have been defined. Suppose the basic dot c is in some V_i for $i \in {}^n\sigma_3$. When trying to classify c on the next level $n+1$, we can only be sure of making the right choice if all c's neighbors and their neighbors (in other words all members of $\mathrm{St}_2(c)$) are classified on level n. We therefore first put:

$$\mathrm{Sec}_n \underset{D}{=} \{c \in V \mid \lg(c) \geq M \wedge \forall d \in \mathrm{St}_2(c) \exists j \in {}^n\sigma_3\, [d \in V_j]\,\}.$$

Sec_n is the decidable set of 'n-secure' basic dots in our classification scheme. (We take $\mathrm{Sec}_0 = ({}^MV)_{\preceq}$ to fulfill (ii) of the lemma.). Now for $i \in {}^n\sigma_3$ define:

$$V_{i\star 0} \underset{D}{=} \Big(\bigcup_{t \in \mathbb{N}} \{c \in {}^tV_i \mid c \in \mathrm{Sec}_n \wedge c \approx {}^tV_{\mathrm{Pre}(i)} \wedge c \overset{2}{\not\approx} {}^tV_{\mathrm{Suc}(i)}\,\}\Big)_{\preceq}.$$
$$V_{i\star 2} \underset{D}{=} \Big(\bigcup_{t \in \mathbb{N}} \{c \in {}^tV_i \mid c \in \mathrm{Sec}_n \wedge c \approx {}^tV_{\mathrm{Suc}(i)} \wedge c \overset{2}{\not\approx} {}^tV_{\mathrm{Pre}(i)}\,\}\Big)_{\preceq}.$$
$$V_{i\star 1} \underset{D}{=} \bigcup_{t \in \mathbb{N}} \{c \in {}^tV_i \mid c \in \mathrm{Sec}_n \wedge c \notin V_{i\star 0} \wedge c \notin V_{i\star 2} \wedge c \# {}^tV_{\mathrm{Pre}(i)} \wedge c \# {}^tV_{\mathrm{Suc}(i)}\,\}.$$

Notice that if for $c \in {}^tV$ we know $c \# {}^tV_{\mathrm{Pre}(i)} \wedge c \# {}^tV_{\mathrm{Suc}(i)}$, then c can still be in $V_{i\star 0}$ or $V_{i\star 2}$, simply because there can be a $d \succ c$ which is in $V_{i\star 0}$ or $V_{i\star 2}$.

The sets $V_{i\star s}$ are decidable and monotone, since $\approx$ is decidable and for $c \in V$ the sets $\mathrm{St}_2(c)$ and $\{b \in V \mid c \prec b\}$ are finite. For $i, j \in {}^{n+1}\sigma_3 \cup \{-1, 3\}$ we see that $j \notin \{\mathrm{Pre}(i), i, \mathrm{Suc}(i)\}$ implies $V_i \# V_j$, and if $i \neq j$ are in ${}^{n+1}\sigma_3$ then $V_i \cap V_j = \emptyset$. This establishes (∗) and (∗∗), and (∗∗∗) follows straight from the definition.

We will use the sets $(V_i)_{i \in \{0,1,2\}^*}$ in a way similar to the proof of lemma (**C**). But other than in that proof, for $n \in \mathbb{N}$ we cannot determine a uniform level

N on V such that for $t \geq N$ all $c \in {}^tV$ are in some V_i with $i \in {}^n\sigma_3$. However, for $x \in \mathcal{V}$ we can use the fans $(W_{x,m})_{m\in\mathbb{N}}$ defined in (**e**) to show that there still is an $N \in \mathbb{N}$ such that x_N is in some V_i with $i \in {}^n\sigma_3$.
This we achieve in the second part of the proof, using a number of claims. We then conclude that the desired morphisms $h_{a,b}$ and $f_{a,b}$ can be derived from our construction above just as in the proof of lemma (**C**).

Part two
We need to prove that for $x \in \mathcal{V}$, for all $n \in \mathbb{N}$ there is $i \in {}^n\sigma_3$ such that $x \in [V_i]$ (which is equivalent to there being an $N \in \mathbb{N}$ with $x_N \in V_i$). Since $V_{\bigcirc_\omega} = V$, this is trivial for $n = 0$.
We use the sets $(\mathrm{Sec}_n)_{n\in\mathbb{N}}$ and a form of double induction, on $n \in \mathbb{N}$ and on $m \in \mathbb{N}$ where m is the index of the fans $(W_{x,m})_{m\in\mathbb{N}}$ which are needed to make the induction step.

The induction basis where $n = 0$ is trivial: clearly for $m, t \in \mathbb{N}$ such that $t \geq M$ we have: $\forall c \in {}^tW^{\wr}_{x,m}\ \exists i \in {}^n\sigma_3\ [c \in V_i]$, since $V_{\bigcirc_\omega} = V$. We now turn to the inductive step going from n to $n+1$. We have to expand our strategy to $W_{x,m+2}$ to ensure that $c \in \mathrm{Sec}_n$ for the relevant $c \in W_{x,m}$, so that these c have no unexpected neighbors and can be classified on level $n+1$.

claim Let $x \in \mathcal{V}$ and $n, m, t \in \mathbb{N}$ such that $t \geq M$ and $\forall c \in {}^tW^{\wr}_{x,m+2} \exists i \in {}^n\sigma_3 [c \in V_i]$. Then: $\exists N \in \mathbb{N}\, \forall c \in {}^NW^{\wr}_{x,m} \exists j \in {}^{n+1}\sigma_3 [c \in V_j]$.

proof By the conditions, $c \in \mathrm{Sec}_n$ for all $c \in ({}^tW^{\wr}_{x,m})_{\preceq}$. For $i \in {}^n\sigma_3$ we see by (**) above: ${}^tV_{\mathrm{Pre}(i)} \# {}^tV_{\mathrm{Suc}(i)}$. Put $A = {}^tV_{\mathrm{Pre}(i)} \cap W_{x,m+2}$ and $B = {}^tV_{\mathrm{Suc}(i)} \cap W_{x,m+2}$, then $A \# B$ are finite subsets of $W_{x,m+2}$, so by our splitting lemma (**B**) there is an $N_i \in \mathbb{N}, N_i \geq t$ such that for all $c, d \in {}^{N_i}W_{x,m+2}$ we have: $(c \approx {}^{N_i}A_{\preceq} \wedge d \approx {}^{N_i}B_{\preceq})$ implies $c \# d$. Again, for all $c \in {}^{N_i}W^{\wr}_{x,m}$ we also have $c \in \mathrm{Sec}_n$.

By definition (**e**) and the choice of A, B we see that for $c \in {}^{N_i}W^{\wr}_{x,m}$ the conditions $c \approx {}^{N_i}A, c \overset{2}{\approx} {}^{N_i}A$ equal the conditions $c \approx {}^{N_i}V_{\mathrm{Pre}(i)}, c \overset{2}{\approx} {}^{N_i}V_{\mathrm{Pre}(i)}$ and likewise the conditions $c \approx {}^{N_i}B, c \overset{2}{\approx} {}^{N_i}B$ equal the conditions $c \approx {}^{N_i}V_{\mathrm{Suc}(i)}, c \overset{2}{\approx} {}^{N_i}V_{\mathrm{Suc}(i)}$.

Therefore for $c \in {}^{N_i}W^{\wr}_{x,m} \cap V_i$ we can decide: `case 0` $c \approx {}^{N_i}V_{\mathrm{Pre}(i)} \wedge c \not\overset{2}{\approx} {}^{N_i}V_{\mathrm{Suc}(i)}$, so $c \in V_{i\star 0}$ OR `case 1` $c \# {}^{N_i}V_{\mathrm{Pre}(i)} \wedge c \# {}^{N_i}V_{\mathrm{Suc}(i)}$, then we must check `case 1.0` there is a $d \succ c$ which is already in $V_{i\star 0}$, then $c \in V_{i\star 0}$ or `case 1.2` there is a $d \succ c$ which is already in $V_{i\star 2}$, then $c \in V_{i\star 2}$ or else `case 1.1` $c \in V_{i\star 1}$ OR `case 2` $c \approx {}^{N_i}V_{\mathrm{Suc}(i)} \wedge c \not\overset{2}{\approx} {}^{N_i}V_{\mathrm{Pre}(i)}$, then $c \in V_{i\star 2}$.

This shows that for all $c \in {}^{N_i}W^{\wr}_{x,m} \cap V_i$ we can find $j \in {}^{n+1}\sigma_3$ with $c \in V_j$. Finding for each $i \in {}^n\sigma_3$ a likewise N_i, put $N = \max(\{N_i \mid i \in {}^n\sigma_3\})$. (end of claim-proof)

claim For all $n, m \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that $\forall c \in {}^N W^{\aleph}_{x,m} \exists i \in {}^n\sigma_3\, [c \in V_i]$.

proof By double induction.
Basis: for $n = 0$ the statement is trivially true for all $m \in \mathbb{N}$.
Induction: Suppose the statement is true for given $n \in \mathbb{N}$ and all $m \in \mathbb{N}$. Then for $m \in \mathbb{N}$, there is $t \in \mathbb{N}$ such that $t \geq M$ and $\forall c \in {}^t W^{\aleph}_{x,m+2}\, \exists i \in {}^n\sigma_3\, [c \in V_i]$. By the previous claim, there is an $N \in \mathbb{N}$ such that $\forall c \in {}^N W^{\aleph}_{x,m} \exists i \in {}^{n+1}\sigma_3\, [c \in V_i]$.
(end of claim-proof)

The claim guarantees that our next construction will define a morphism. We use the sets V_i to define a canonical function $h_{a,b}$ from V to $\{0,1,2\}^*$, quite similar to the proof of lemma (**C**). For $n \in \mathbb{N}$ and $c \in {}^n V$ put:

$$h_{a,b}(c) \underset{D}{=} \text{the unique } i \text{ in } \{0,1,2\}^* \text{such that: } \lg(i) \leq \lg(c) = n \text{ and } c \in V_i \text{ and}$$
$$\forall j \in \{0,1,2\}^*\, [(\lg(j) \leq n \wedge c \in V_j) \rightarrow i \preceq_\omega j].$$

That this definition is valid is easily seen, since there are only finitely many decidable conditions to check and c is certainly in $V_{\bigcirc_\omega}$.

claim $h_{a,b}$ is a $\preceq$-morphism from $\mathcal{V}$ to $\sigma_{3,\text{ter}}$.

proof It follows from $(\ast\ast\ast)$ above that for $c \preceq d \in V$ we have $h_{a,b}(c) \preceq h_{a,b}(d)$. Let $x \in \mathcal{V}$, we need to show that $h_{a,b}(x)$ is a point in σ_3. This follows easily however from the claim above, since we see that for all $n \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that $h_{a,b}(x_N) \in {}^n\sigma_3$. Finally, suppose $y \in \mathcal{V}$ is a point such that $h_{a,b}(y) \#_{\mathbb{R}} h_{a,b}(x)$, then we must show $y \# x$. Since $h_{a,b}(y) \#_{\mathbb{R}} h_{a,b}(x)$, we can determine $n \in \mathbb{N}$ and $i, j \in {}^n\sigma_3$ such that $i \#_{\mathbb{R}} j$ and $h_{a,b}(y) \prec i$ and $h_{a,b}(x) \prec j$. However, $i \#_{\mathbb{R}} j$ equals $j \notin \{\mathrm{Pre}(i), i, \mathrm{Suc}(i)\}$. There is $s \in \mathbb{N}$ such that $y_s \in V_i$ and $x_s \in V_j$ where $V_i \# V_j$ since $j \notin \{\mathrm{Pre}(i), i, \mathrm{Suc}(i)\}$ (by $(\ast\ast)$), so we see that $y_s \# x_s$ and so $y \# x$. (end of claim-proof)

We define: $f_{a,b} \underset{D}{=} f_{\text{evl},3} \circ h_{a,b}$ (canonically since $h_{a,b}$ and $f_{\text{evl},3}$ are constructed canonically). Clearly $f_{a,b}|_{W_a} \equiv_{\mathbb{R}} 0$ and $f_{a,b}|_{W_b} \equiv_{\mathbb{R}} 1$, proving (i) of the lemma.

The only thing left to prove is that if $c \in {}^M W$ with $a \# c \# b$ then $f_{a,b}|_{W_c} \subseteq [\frac{1}{3}, \frac{2}{3}]$. Yet it follows from our construction that $c \in V_1$. Also trivially $f_{a,b}|_{V_1} \subseteq [\frac{1}{3}, \frac{2}{3}]$.
(END OF PROOF)

THEOREM: (from 4.0.8) Every star-finitary natural space is metrizable.

PROOF: It suffices to prove the theorem for our star-finite $\propto$-spread $(\mathcal{V}, \mathcal{T}_\#)$.

claim $(\mathcal{V},\mathcal{T}_\#)$ meets the conditions of the Urysohn metrization lemma (**A**).

proof We need to show:

(i) For all $n\in\mathbb{N}$, for all $a\#b\in{}^nV^\bullet$ there is a canonical morphism $f_{a,b}$ from $\mathcal{V}^\bullet$ to $[0,1]$, such that $f_{a,b}|_{V_a^\bullet}\equiv_{\mathbb{R}}0$ and $f_{a,b}|_{V_b^\bullet}\equiv_{\mathbb{R}}1$.

(ii) For all $n\in\mathbb{N}$ and $a\#b\in{}^nV^\bullet$: if $c\in{}^nV^\bullet$ with $a\#c\#b$ then $f_{a,b}|_{V_c^\bullet}\subseteq[\frac{1}{3},\frac{2}{3}]$

(iii) For any $\#$-open $\mathcal{U}\subseteq\mathcal{V}$ and successor point $x\in\mathcal{U}$ there is an $n\in\mathbb{N}$ such that for all $a\in{}^nV$ we have: $a\approx x_n$ implies $[\![a]\!]\subseteq\mathcal{U}$.

Ad (i) and (ii): Consider that $(\mathcal{V}^\bullet,\mathcal{T}_\#)$ is again a star-finite $\propto$-spread. Now apply the Urysohn function lemma (**E**) above.
Ad(iii): this is precisely the content of lemma (**D**) above. (end of claim-proof)

This shows that $(\mathcal{V},\mathcal{T}_\#)$ satisfies the requirements of the Urysohn metrization lemma (**A**), and so $(\mathcal{V},\mathcal{T}_\#)$ is metrizable. (END OF PROOF)

A.3.19 Proof of meta-theorem 4.2.4

The two-player game LIfE serves as an illustration of our belief that INT can be (formally) interpreted in RUSS also, in an elegant way. We state and prove:

META-THEOREM: (repeated from 4.2.4)

(i) In LIfE, we can prove **CP**.

(ii) Suppose GoD is omniscient. Then we can prove $\neg\exists B\subset\mathbb{N}^*[B$ is a non-inductive bar on $\mathcal{N}]$ for LIfE.

(iii) Given enough time, HuMaN can discover that by an overwhelming odds ratio, GoD plays only recursive sequences.

COROLLARY: In CLASS, we can prove **CP** and **BT** for the game LIfE.

PROOF: Ad (i). Suppose that HuMaN has a set $A\subseteq\mathbb{N}^{\mathbb{N}}\times\mathbb{N}$ such that in LIfE: $\forall\alpha\in\mathbb{N}^{\mathbb{N}}\exists n\in\mathbb{N}[(\alpha,n)\in A]$. This means: for any sequence α played by GoD, HuMaN can produce at a finite moment in time an $n\in\mathbb{N}$ such that $(\alpha,n)\in A$.

Now let $\alpha\in\mathbb{N}^{\mathbb{N}}$. Let GoD play the sequence γ which mimicks α, starting out as $\gamma=\alpha(0),\alpha(1),\ldots$ without revealing any information to HuMaN. Say at point m in time (when GoD has revealed precisely $\overline{\alpha}(m)$, the first m values of α), HuMaN produces n such that $(\gamma,n)\in A$. By the rules of LIfE, player GoD may switch to any β (computable, but HuMaN does not know this) such that

$\overline{\beta}(m) = \overline{\gamma}(m) = \overline{\alpha}(m)$ to conclude that for any such β we have $(\beta, n) \in A$. Therefore $\forall \alpha \in \mathbb{N}^{\mathbb{N}} \exists m, n \in \mathbb{N} \forall \beta \in \mathbb{N}^{\mathbb{N}} [\overline{\alpha}(m) = \overline{\beta}(m) \to (\beta, n) \in A]$, which proves **CP** for LIfE.

Ad (ii). We use the correspondence between natural Baire space $\mathcal{N}$ and $\mathbb{N}^{\mathbb{N}}$, described in 2.0.2. Suppose HuMaN has a set $B \subset \mathbb{N}^*$ which is non-inductive (meaning it does not descend from a genetic bar on $\mathcal{N}$), and a bar $C \subseteq \mathbb{N}^*$ on $\mathcal{N}$ with $B \subseteq C$. Since B is non-inductive, with omniscience GoD can play a sequence $\alpha = p^*$ where $\overline{\alpha} = p \in \mathcal{N}$ is given by $p = p_0, p_1, \ldots$ (with $p_0 = \bigcirc$ and $p_{i+1} \propto p_i$) and where $(p_i)_{\preceq} \cap B$ is non-inductive on $\mathbb{N}^*_{p_i} = (p_i)_{\preceq}$ for all i up until and including the n for which $\overline{\alpha}(n) = p_n \in C$ (since C is claimed to be a bar, HuMaN must produce such n at some finite point in time).

Having received from HuMaN the n for which $\overline{\alpha}(n) = p_n \in C$, GoD switches to the recursive sequence $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $\beta = p_n \star \underline{0}$. To arrive at β, GoD can claim to have used the recursive sequences $p_1 \star \underline{0}, p_2 \star \underline{0}, \ldots$ and to have switched $n-1$ times, therefore GoD has played by the rules.

We see now, that $\overline{\alpha}(n) = p_n$ is in C but not in B, since $p_n \in B$ would imply that $(p_n)_{\preceq} \cap B$ is inductive on $\mathbb{N}^*_{p_n} = (p_n)_{\preceq}$, contradicting the above. We conclude: $B \neq C$. This proves (ii).

Ad (iii). This is basically the same as the physical experiment described in [Waa2005], section 7. HuMaN can build a covering of the recursive unit interval with a sequence of rational intervals $(R_n)_{n \in \mathbb{N}}$ such that the sum of the lengths of these intervals does not exceed 2^{-40}. Asking GoD for a non-recursive sequence $\alpha \in [0, 1]$, HuMaN will eventually discover that α falls within one of the intervals R_n. (GoD can only switch finitely many times, and then in the end is stuck with a recursive α, which will eventually be captured by some R_n.) The odds of this happening for a non-recursive α are overwhelmingly small, proving (iii).

The corollary follows from (i) and (ii), since with classical logic GoD is omniscient and $\neg \exists B \subset \mathbb{N}^* [B$ is a non-inductive bar on $\mathcal{N}]$ is equivalent to ‘every bar on $\mathcal{N}$ is inductive’. (END OF PROOF)

A.4 CONSTRUCTIVE CONCEPTS AND AXIOMS USED

A.4.0 Basic axioms and concepts In this section we present some axioms and concepts described in the literature, pertaining to constructive mathematics in particular (INT, RUSS, and BISH). Definitions given in this section may slightly differ from similar earlier definitions given for natural spaces, to conform to standard practice in INT.

A relatively short discussion of most of these axioms, and comparisons of interrelative strength, can be found under the same names in [Waa2005]. More fundamental discussions on intuitionistic axioms are to be found in [Vel1981], [Vel2008], [Vel2009], and [Vel2011]. More fundamental discussions on a large number of constructive axioms (and comparisons of interrelative strength) are to be found in the standard works [Bee1985] and [Tro&vDal1988].

A.4.1 Constructive logic is intuitionistic logic The common practice in constructive mathematics is to use intuitionistic logic. We explain what we mean by describing the meaning of our quantifiers and expressions involving them. Before we continue, let us state that certain mathematical notions will be taken as *primitive*, that is: hopefully understood but not defined in terms of other notions. One of these notions is the notion of a *sequence*, for instance a sequence of natural numbers.

We call $0, 1, 2, \ldots$ a sequence of natural numbers. There are many other such sequences of course, for instance the sequence of prime numbers $2, 3, 5, \ldots$ The set of all sequences of natural numbers is often called $\mathbb{N}^{\mathbb{N}}$. In intuitionism the tradition is however to call this set σ_ω, for reasons which have hopefully become apparent in our previous narrative.

Cantor's diagonal argument shows that we cannot produce all sequences of natural numbers one after the other. This exhibits an important difference between σ_ω and $\mathbb{N}$. For we do have a way to produce all natural numbers, one after the other, even if we are never done with $\mathbb{N}$ as a whole. But to produce just one element of σ_ω is as much work as producing all of $\mathbb{N}$.

Other primitive notions are those of a 'set', a 'subset' and an 'element' of a subset, along with the notion of a 'collection of subsets'. We write $\emptyset$ for the empty subset.

Primitive notions relating to our use of quantifiers are the notion of 'method' and 'existence'. We say that a mathematical object such as a natural number, a sequence of natural numbers, or a subset with a certain property P *exists* if and only if we have a method to construct it.
Then we write, for example: $\exists\alpha\in\sigma_\omega\,[\mathrm{P}(\alpha)]$ (*there is* an α in σ_ω with property P). If P is a property applicable to sequences of natural numbers, then we can form the subset $\{\alpha\in\sigma_\omega \mid \mathrm{P}(\alpha)\}$. If P is a property applicable to natural numbers and n is in $\mathbb{N}$, then we write $n=\mu s\in\mathbb{N}\,[P(s)]$ to mean that n is the smallest natural number with property P.

We abbreviate 'if and only if' with 'iff'.

We assume the reader is familiar with the logical symbols $\forall$, $\exists$, $\exists!$, $\wedge$, $\vee$, $\neg$ and $\to$. We often use them to abbreviate otherwise lengthy statements. Let P be a property applicable to the elements of a set or collection X. '$\forall x\in X\,[P(x)]$' means: for all x in X we can prove $P(x)$. '$\exists x\in X\,[P(x)]$' means: there is an x in X such that $P(x)$, as explained above. '$\exists!x\in X\,[P(x)]$' means: there is an x in X such that $P(x)$ and for all y in X: if $P(y)$ then $y=x$.
If on the other hand P and Q are statements, then '$P\wedge Q$' means: we can prove both P and Q. '$P\vee Q$' means: we can choose either P or Q, and prove the chosen statement. So in fact '$P\vee Q$' is equal to: '$\exists s\in\{0,1\}\,[(s=0\wedge P)$ or $(s=1\wedge Q)]$'. '$P\to Q$' means: P implies Q (we can prove Q from P). Finally, '$\neg P$' means that we can prove a contradiction from P (and our axioms).

We have to distinguish between P and $\neg\neg P$. Of course $\neg\neg P$ follows from P, but in general the knowledge that $\neg P$ is impossible does not supply us with a proof of P. Similarly we distinguish between $\neg\forall x\in X\,[P(x)]$ and $\exists x\in X\,[\neg P(x)]$. (There are situations in INT in which we can prove both $\neg\forall x\in X\,[P(x)]$ and $\neg\exists x\in X\,[\neg P(x)]$.).

A.4.2 Functions are Cartesian subsets

Having taken the notion of 'set' and 'subset' as primitive, we define a function from an apartness space $(V, \#_1)$ to another apartness space $(W, \#_2)$ as a subset of the cartesian product $(V\times W, \#_\pi)$ (see def. 3.5.1) such that:

1. for all $x\in V$ there is a $y\in W$ such that $(x,y)\in f$
2. for all $x, v\in V$ and $y,z\in W$: if $(x,y)\in f$ and $(v,z)\in f$ and $y\,\#_2\,z$ then $x\,\#_1\,v$.

Then for any pair $(x,y)\in f$ we write: $f(x)\equiv y$ or $f(x)=y$.

The constructive interpretation of the quantifiers 'for all' and 'there is' ensures in our eyes that this definition nicely captures the connotation of methodicity which the word ‘function’ carries. In this book we almost always work with morphisms anyway, but the definition above is strictly speaking necessary to underpin the theorems on representability of continuous functions by morphisms.

A.4.3 Some intuitionistic definitions To define the relevant intuitionistic axioms of (continuous) choice we need a number of straightforward definitions, which closely resemble our earlier definitions regarding natural spaces. The reader should take the definitions below as intuitionistic parallels.

DEFINITION: (in INT) Let σ_ω denote the universal spread of all infinite sequences of natural numbers ($\sigma_\omega = \mathbb{N}^{\mathbb{N}}$). Write $\overline{\overline{\sigma}}_\omega$ for the set of finite sequences of natural numbers (often written like this: $\overline{\overline{\sigma}}_\omega = \mathbb{N}^*$). For α in σ_ω we write $\overline{\alpha}(n)$ for the finite sequence $\alpha(0), \ldots, \alpha(n-1)$ formed by the first n values of α. Then $\overline{\alpha}(n) \in \overline{\overline{\sigma}}_\omega$, and vice versa $\overline{\overline{\sigma}}_\omega = \{\overline{\alpha}(n) \mid \alpha \in \sigma_\omega, n \in \mathbb{N}\}$. A subset B of $\overline{\overline{\sigma}}_\omega$ is called *decidable* iff for all $a \in \overline{\overline{\sigma}}_\omega$ we have a finite decision procedure to determine whether $a \in B$ or $a \notin B$. A subset B of $\overline{\overline{\sigma}}_\omega$ is a *bar* on a subset A of σ_ω iff $\forall \alpha \in A \, \exists n \in \mathbb{N} \, [\, \overline{\alpha}(n) \in B \,]$, and a *thin bar* iff $\forall \alpha \in A \, \exists ! n \in \mathbb{N} \, [\, \overline{\alpha}(n) \in B \,]$

Now let a be in $\overline{\overline{\sigma}}_\omega$, then a is a finite sequence of natural numbers. We write $lg(a)$ for the length of this finite sequence. So if $a = a_0, \ldots, a_{n-1}$ then $lg(a) = n$. There is a sequence of length 0, namely the *empty sequence* denoted by $\langle\!\rangle$. For $i < lg(a)$ we then write a_i for the i^{th} element of this finite sequence. If $a = a_0, a_1, \ldots, a_{lg(a)-1}$ and $b = b_0, b_1, \ldots, b_{lg(b)-1}$ are in $\overline{\overline{\sigma}}_\omega$ then we write $a \star b$ for the *concatenation* $a_0, a_1, \ldots, a_{lg(a)-1}, b_0, b_1, \ldots, b_{lg(b)-1}$ of a and b. We write $a \sqsubseteq b$ iff there is a c in $\overline{\overline{\sigma}}_\omega$ such that $b = a \star c$, and we write $a \sqsubset b$ iff in addition $lg(b) > lg(a)$.

A function f from σ_ω to $\mathbb{N}$ is called a *spread-function* iff there is a function g from $\overline{\overline{\sigma}}_\omega$ to $\mathbb{N}$ such that for each α in σ_ω: $\exists ! n \in \mathbb{N} \, [g(\overline{\alpha}(n)) > 0]$ and moreover for all $n \in \mathbb{N}$: $g(\overline{\alpha}(n)) > 0 \to f(\alpha) = g(\overline{\alpha}(n)) - 1$.[24] More generally a function f from σ_ω to σ_ω is called a *spread-function* iff there is a function g from $\overline{\overline{\sigma}}_\omega$ to $\overline{\overline{\sigma}}_\omega$ such that for each α in σ_ω and $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that: $\overline{f(\alpha)}(n) = g(\overline{\alpha}(m))$, and moreover $g(\overline{\alpha}(n)) \sqsubseteq g(\overline{\alpha}(n+1))$. (END OF DEFINITION)

REMARK: Spread-functions from σ_ω to σ_ω correspond one-on-one to natural morphisms from $\mathcal{N}$ to $\mathcal{N}$. We have declined in this monograph to define natural ‘morphisms’ from σ_ω to $\mathbb{N}$, but this is easily remedied. (END OF REMARK)

[24] Notice that $\{a \in \overline{\overline{\sigma}}_\omega \mid g(a) > 0\}$ is a decidable thin bar. This shows that the concept of spread-function is inherently the same as the concept of a decidable (thin) bar.

A.4.4 **Axioms of continuous choice in INT** The fundamental intuitionistic axiom of continuous choice $\mathbf{AC}_{11}$ can now be formulated as follows:

$\boxed{\mathbf{AC}_{11}}$ Let A be a subset of $\sigma_\omega \times \sigma_\omega$ such that:

($\star$) $\forall \alpha \in \sigma_\omega \exists \beta \in \sigma_\omega [(\alpha, \beta) \in A]$

Then there is a spread-function γ from σ_ω to σ_ω such that for each α in σ_ω: $(\alpha, \gamma(\alpha))$ is in A. We say that γ *fulfills* ($\star$).

We formulate four weaker versions of this axiom: $\mathbf{AC}_{10}$, **CP**, $\mathbf{AC}_{01}$, and $\mathbf{AC}_{00}$. The last two are simple axioms of countable choice, whereas $\mathbf{AC}_{10}$ is still an axiom of continuous choice, also known as 'Brouwer's principle for numbers'. $\mathbf{AC}_{10}$ implies the so-called continuity principle **CP**. We do not defend the axioms here since they are broadly discussed in the literature (see e. g. [Kle&Ves1965], [GiSwVe1981], [Vel1981] and [Tro&vDal1988]). We begin with the weaker axioms dealing with continuous choice:

$\boxed{\mathbf{AC}_{10}}$ Let A be a subset of $\sigma_\omega \times \mathbb{N}$ such that:

($\star\star$) $\forall \alpha \in \sigma_\omega \exists n \in \mathbb{N} [(\alpha, n) \in A]$

Then there is a spread-function γ from σ_ω to $\mathbb{N}$ such that for each α in σ_ω: $(\alpha, \gamma(\alpha))$ is in A. We say that γ *fulfills* ($\star\star$).

$\boxed{\mathbf{CP}}$ Let A be a subset of $\sigma_\omega \times \mathbb{N}$ such that:

$\forall \alpha \in \sigma_\omega \exists n \in \mathbb{N} [(\alpha, n) \in A]$

Then: $\forall \alpha \in \sigma_\omega \exists n \in \mathbb{N} \exists m \in \mathbb{N} \forall \beta \in \sigma_\omega [\overline{\alpha}(m) = \overline{\beta}(m) \rightarrow (\beta, n) \in A]$.

A.4.5 **Axioms of countable choice in BISH** We present two simple axioms of *countable choice* in decreasing order of strength:

$\boxed{\mathbf{AC}_{01}}$ Let A be a subset of $\mathbb{N} \times \sigma_\omega$ such that:

($*$) $\forall n \in \mathbb{N} \exists \alpha \in \sigma_\omega [(n, \alpha) \in A]$

Then there is a function h from $\mathbb{N}$ to σ_ω such that for each $n \in \mathbb{N}$: $(n, h(n))$ is in A. We say that h *fulfills* ($*$).

$\boxed{\mathbf{AC}_{00}}$ Let A be a subset of $\mathbb{N} \times \mathbb{N}$ such that:

($**$) $\forall n \in \mathbb{N} \exists m \in \mathbb{N} [(n, m) \in A]$

Then there is a function h from $\mathbb{N}$ to $\mathbb{N}$ such that for each $n \in \mathbb{N}$: $(n, h(n))$ is in A. We say that h *fulfills* ($**$).

A.4.6 **Axioms of dependent choice in BISH** Likewise we present two axioms of *dependent choice* in decreasing order of strength. For an intuitionistic justification of these axioms we refer the reader to [Waa1996].

$\boxed{\mathbf{DC}_1}$ Let δ be in σ_ω, and let A be a subset of σ_ω. Suppose R is a subset of $A\times A$ such that:

$\delta\in A \wedge \forall\alpha\in A\ \exists\beta\in A\,[\,(\alpha,\beta)\in R\,]$

Then there is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of σ_ω such that $\gamma_0=\delta$ and for each $n\in\mathbb{N}$: (γ_n,γ_{n+1}) is in R.

$\boxed{\mathbf{DC}_0}$ Let $s\in\mathbb{N}$, and let A be a subset of $\mathbb{N}$. Suppose R is a subset of $A\times A$ such that:

$s\in A \wedge \forall n\in A\ \exists m\in A\,[\,(n,m)\in R\,]$

Then there is an α in σ_ω such that $\alpha(0)=s$ and for each $n\in\mathbb{N}$: $(\alpha(n),\alpha(n+1))$ is in R.

A.4.7 **Axiom of extensionality** The axiom of extensionality states that we do not distinguish between infinite sequences which are termwise identical, even though they may have different descriptions/definitions.

$\boxed{\mathbf{Ext}}$ Let $\alpha,\beta\in\sigma_\omega$ such that $\forall n\in\mathbb{N}[\alpha(n)=\beta(n)]$. Then $\alpha=\beta$.

This axiom is often adopted rather silently (like we do also). In RUSS, the different algorithms describing infinite sequences play an important role, but still one wishes to see infinite sequences themselves as equal when they are termwise identical.

A.4.8 **Bar induction, Brouwer's Thesis and the Fan Theorem** To phrase the principle of Bar Induction for Decidable bars ($\mathbf{BI_D}$) we need:

DEFINITION: A subset A of $\overline{\overline{\sigma}}_\omega$ is called *downwards inductive*[25] iff for all α in $\overline{\overline{\sigma}}_\omega$: $\forall n\in\mathbb{N}\,[\alpha\star n\in A]\rightarrow\alpha\in A$. (END OF DEFINITION)

$\boxed{\mathbf{BI_D}}$ Let B be a decidable bar on σ_ω. Suppose A is a downwards inductive subset of $\overline{\overline{\sigma}}_\omega$ such that $B\subseteq A$. Then the empty sequence $\langle\!\langle\,\rangle\!\rangle$ (of length 0) is in A.

[25] We must distinguish from the already defined notion 'inductive'.

REMARK: In classical mathematics $\mathbf{BI_D}$ can be derived from the principle of the excluded middle. The above version of the bar theorem is therefore classically true. In [Waa2005] $\mathbf{BI_D}$ is derived from **BT** (see 2.4.1) which also holds both in CLASS and INT. (END OF REMARK)

One of the results following from $\mathbf{BI_D}$ is the axiom known as the fan theorem **FT**. We need a preliminary definition.

DEFINITION: Let σ_2 be the binary fan $(\{0,1\}^{\mathbb{N}})$. Write $\overline{\overline{\sigma}}_2$ for $\{0,1\}^*$, the set of finite sequences of elements of $\{0,1\}$. Then $\overline{\overline{\sigma}}_2 = \{\overline{\alpha}(n) \mid \alpha \in \sigma_2,\ n \in \mathbb{N}\}$. A subset B of $\overline{\overline{\sigma}}_2$ is a *bar* on σ_2 iff $\forall \alpha \in \sigma_2 \exists n \in \mathbb{N} [\overline{\alpha}(n) \in B]$. (END OF DEFINITION)

$\boxed{\mathbf{FT}}$ If B is a decidable bar on σ_2, then B contains a finite bar on σ_2 (in other words: then $\exists n \in \mathbb{N} \forall \alpha \in \sigma_2 \exists m < n [\overline{\alpha}(m) \in B]$).

A.4.9 Basic axioms in RUSS: Church's Thesis The basic axiom in RUSS is of course Church's Thesis: 'every sequence of natural numbers is given by a recursive rule' (many results in RUSS already follow from the weaker statement: 'the set of partial functions from $\mathbb{N}$ to $\mathbb{N}$ is countable'). A partial recursive function α from $\mathbb{N}$ to $\mathbb{N}$ will usually be denoted by something like 'ϕ_e' where the natural number e is the *recursive index* of α. This recursive index is nothing but the encoding of the finite algorithm which for each $n \in \mathbb{N}$ tries to compute $\alpha(n)$. There is a decidable subset $I(\mathbb{N},\mathbb{N})$ of $\mathbb{N}$ such that each e in $I(\mathbb{N},\mathbb{N})$ is a properly formed recursive index of a partial recursive function from $\mathbb{N}$ to $\mathbb{N}$, and vice versa for each partial recursive function α from $\mathbb{N}$ to $\mathbb{N}$ there is an e in $I(\mathbb{N},\mathbb{N})$ such that $\forall n \in \mathbb{N} [\alpha(n) \doteq \phi_e(n)]$, where '$\doteq$' stands for: 'equal if one of the algorithms terminates, given the input'.

It turns out one can canonically encode each finite recursive computation as a natural number, see [Kle1952]. This is the basis of Kleene's decidable T-predicate on triples of natural numbers (e,n,k), given by:

$T(e,n,k) \iff e$ is a recursive index and k is the canonical encoding of the computation of $\phi_e(n)$.

If $T(e,n,k)$, then the algorithm ϕ_e terminates on the input n. But we are mostly interested in the result of the computation k, and in its length (the number of canonical subcomputations leading to the result). Both can be canonically derived from k of course, using recursive functions Outc and Lgth. So if $T(e,n,k)$, then $\phi_e(n) = \mathrm{Outc}(k)$ and the length of the computation k equals $\mathrm{Lgth}(k)$.

In this terminology we formulate the axiom Church's Thesis[26] thus:

$\boxed{\textbf{CT}}$ $\forall \alpha \in \sigma_\omega \exists e \in I(\mathbb{N},\mathbb{N}) \forall n \in \mathbb{N} [\alpha(n) = \phi_e(n)]$

If $\forall n \in \mathbb{N} [\alpha(n) = \phi_e(n)]$ for $\alpha \in \sigma_\omega$ and $e \in I(\mathbb{N},\mathbb{N})$, then in particular we have: $\forall n \in \mathbb{N} \exists k \in \mathbb{N} [T(e,n,k)]$. The set $\textbf{TOT} = \{e \in I(\mathbb{N},\mathbb{N}) \mid \forall n \in \mathbb{N} \exists k \in \mathbb{N} [T(e,n,k)]\}$ therefore plays an important role in RUSS.

The combination of **CT** with $\textbf{AC}_{00}$ is equivalent to an axiom known as $\textbf{CT}_0$, which forms a connection between **CT** and choice axioms:

$\boxed{\textbf{CT}_0}$ Let A be a subset of $\mathbb{N} \times \mathbb{N}$ such that:

$\forall n \in \mathbb{N} \exists m \in \mathbb{N} [(n,m) \in A]$

Then there is a *recursive* function h from $\mathbb{N}$ to $\mathbb{N}$ such that for each $n \in \mathbb{N}$: $(n, h(n))$ is in A.

From $\textbf{CT}_0$ we can derive a more complex choice axiom $\textbf{CT}_{01}$, which plays a part in the defense of $\textbf{CT}_{11}$ in [Waa2005]:

$\boxed{\textbf{CT}_{01}}$ Let A, B be subsets of $\mathbb{N} \times \mathbb{N}$, where B is *decidable*, such that:

$\forall n \in \mathbb{N} [\exists y \in \mathbb{N} [(n,y) \in B] \rightarrow \exists m \in \mathbb{N} [(n,m) \in A]]$

Then there is a *partial recursive* function h from $\mathbb{N}$ to $\mathbb{N}$ such that for each $n \in \mathbb{N}$: if $\exists y \in \mathbb{N} [(n,y) \in B]$ then $h(n)$ is defined and $(n, h(n))$ is in A.

$\textbf{CT}_{01}$ follows from $\textbf{CT}_0$. $\textbf{CT}_{01}$ is the first step to an even broader choice axiom known as 'Extended Church's Thesis' ($\textbf{ECT}_0$), which is widely accepted in RUSS. But the phrasing of $\textbf{ECT}_0$ and its defense (at least in [Tro&vDal1988]) are in logical terms and do not appeal to the author. We present a simpler version $\textbf{CT}_{11}$ for which an intuitive defense is given in [Waa2005].[27]

$\boxed{\textbf{CT}_{11}}$ Let A be a subset of $\mathbb{N} \times \mathbb{N}$ such that

$\forall n \in \textbf{TOT} \exists m \in \mathbb{N} [(n,m) \in A]$.

Then there is a partial recursive function h from $\mathbb{N}$ to $\mathbb{N}$ such that for all $n \in \textbf{TOT}$ we have: $(n, h(n)) \in A$.

[26] Originally 'Church's Thesis' stands for the idea that any 'mechanically' obtainable sequence must be computable by a Turing machine.

[27] We try to adhere to the principle that mathematics needs clear axioms which represent our mathematical intuition (not merely serve mathematical convenience). Some technical axioms however have practical advantages for comparison purposes.

A.4.10 Markov's Principle A second important axiom in RUSS is called Markov's Principle: 'if it is impossible that a total recursive function α does not achieve the value 1 for some n in $\mathbb{N}$, then there is an n in $\mathbb{N}$ with $\alpha(n)=1$'. Formally:

MP Let α be in σ_ω such that $\neg\neg\exists n\in\mathbb{N}[\alpha(n)=1]$. Then $\exists n\in\mathbb{N}[\alpha(n)=1]$.

In [Waa2005] a plea is given to consider adopting **MP** in intuitionism as well. Joan Moschovakis ([Mos2012]) holds a similar view.

A.4.11 Axiom of induction We present the principle of induction as an axiom.

Ind Let A be a subset of $\mathbb{N}$ such that $0\in A$ and for all $n\in\mathbb{N}$: $n\in A$ implies $n+1\in A$. Then $A=\mathbb{N}$, that is: $n\in A$ for all $n\in\mathbb{N}$.

A.4.12 Axiom of decidable-bar descent for INT, RUSS and CLASS We phrase a Lindelöf-type axiom which holds in CLASS, INT and RUSS.

DEFINITION: Let B and C be two bars on σ_ω, then B *descends* from C iff for all c in C there is a b in B such that $b\sqsubseteq c$. (END OF DEFINITION)

BDD Every bar on σ_ω descends from a decidable bar on σ_ω.

The proof of **BDD** from **BT** is immediate (CLASS and INT), whereas the proof of **BDD** from $\mathbf{CT}_{11}$ in RUSS is given in [Waa2005], using a result from [Ish1993]. In [Waa2005], **BT** is shown to be equivalent to the combination of $\mathbf{BI_D}$ and **BDD**. **BDD** is easily seen to be equivalent to:

BDD* Every bar on σ_ω descends from a decidable thin bar on σ_ω.

A.4.13 More axioms More relevant axioms can be found in the literature, notably [Tro&vDal1988], [Bee1985], [Ish2006] and [Vel2011]. In [Waa2005], apart from the above some other interesting axioms and axiomatics relating to constructive mathematics are discussed as well.

A.5 ADDITIONAL REMARKS

A.5.0 **Containment and refinement** Notice that in the case of dots being rational intervals, the refinement relation can be defined completely in terms of a natural containment relation $\subseteq$ derived from the pre-apartness relation as follows: $a \subseteq b$ iff $c \# b$ implies $c \# a$ for all $c \in V$. In many of our interest spaces, a similar approach yields a decidable containment relation. But in general this approach involves checking an infinite number of conditions, which leaves the so-defined containment relation non-decidable and therefore unwieldy for practical purposes.

A.5.1 **Classical treatment of equivalence** The usual classical approach is to work with equivalence classes as the resulting points (a real number usually is defined as the equivalence class of a Cauchy-sequence of rational numbers). In practice this is cumbersome, since all computations on equivalence classes require working with the representatives of these classes. It is therefore more efficient to work with the original sequences and the original apartness relation directly. For theorists who so desire, the translation to equivalence classes is simple, since all definitions will respect the apartness-induced equivalence relation. In topological terms our approach means a both practical and foundational way of dealing with a quotient space of Baire space with respect to a Π^1_0-equivalence relation.

A.5.2 **Details of proving proposition 1.2.2** In the proof of proposition 1.2.2, the isomorphisms g, h between $(\mathcal{V}, \mathcal{T}_{\#})$ and $(\mathcal{W}, \mathcal{T}_{\#_2})$ are $\preceq$-morphisms. This is without loss of generality, since by 1.1.6 we can always lift arbitrary isomorphisms g, h to $\preceq$-isomorphisms g', h' between $(\mathcal{V}^{\wr}, \mathcal{T}^{\wr}_{\#})$ and $(\mathcal{W}^{\wr}, \mathcal{T}^{\wr}_{\#_2})$, where the latter is also basic-open trivially. Thus establishing the proposition for $(\mathcal{V}^{\wr}, \mathcal{T}^{\wr}_{\#})$ with $f' = h' \circ g'$, we can use the $\wr$-automorphism $f = \mathbf{id}_* \circ f'$ to establish the proposition for $(\mathcal{V}, \mathcal{T}_{\#})$.

A.5.3 **Lazy convergence and isolated points** A main theme in defining natural spaces and morphisms is 'lazy convergence'. Points may themselves choose, so to say, when a next real step in the refinement takes place. However, if one is not careful this leads to some rather unexpected issues with isolated points (points which as a one-point set are open in $\mathcal{T}_{\#}$). To avoid these issues, we have sharpened the definition of 'trail space' and '(in)finite product' given in the first edition of this monograph.

A.5.4 **Spaces which cannot be represented as a natural space** Our discussion in 2.4.2 shows that the 'space' of all Baire morphisms cannot be represented as a natural space. This is a frequently occurring theme for function spaces. It would seem to be highly relevant to study various ways in which we can work with function spaces which do not allow a representation as a natural space. It also seems highly worthwhile to find natural representations for various classes of function spaces. We give one example (already discussed by Brouwer) in the examples' section A.2.4 above.

One way of dealing with a function space F which cannot be represented as a natural space, is to construct a natural space of which F forms a topological subspace. This is often not really satisfying, but still allows to use the functions as 'separably countable' objects. But we leave this for further research, and what has already been done in the constructive literature on function spaces. The issue is discussed also in chapter zero of [Waa1996], where some (most likely not the best) examples and results are given.

A.5.5 **Definition of spreads and spraids** In the definition (2.1.2) of 'spread' we exact that each infinite $\prec$-trail defines a point. This is primarily to obtain a precise match with Brouwer's intuitionistic spreads. Most things seem to work fine without the condition, and we obtain the same inductive spraids with or without. It is however quite instructive to see the effect of this condition on the (in)finite-product definition. For example, if we take the simple finite product $\Pi_{i\leq 1}\sigma_{\mathbb{R}}$ (which represents $\mathbb{R}^2$, see def. 3.5.1) then the infinite $\prec_\pi$-trail $(([0, 2^{-n}], \bigcirc_{\mathbb{R}})_{n\in\mathbb{N}})$ does not define a point. So with the condition, the simple (in)finite-product definition almost never yields a spread even when its constituents are all spreads. Without the condition, the simple (in)finite-product of spreads is again a spread, but now the problem is transferred to the inductiveness of the spreads involved (again use $\Pi_{i\leq 1}\sigma_{\mathbb{R}}$).

For elegance we wish to retain the simple infinite-product for natural spaces in general. This means that for spreads $((\mathcal{V}_n, \mathcal{T}_{\#_n}))_{n\in\mathbb{N}}$ we need the sharper finite products $\Pi^{\sigma}_{i\leq n}(\mathcal{V}_i, \mathcal{T}_{\#_i})$ and infinite product $\Pi^{\sigma}_{n\in\mathbb{N}}(\mathcal{V}_n, \mathcal{T}_{\#_n})$ defined in 3.5.1.

Matching Brouwer's spreads precisely is important to us so we adopt the condition. The above is an illustration of how deeply linked most definitions in this monograph are.

A.5.6 **Definition of inductive open covers from 3.1.0** The definition of pointwise inductive open covers in 3.1.0 is trickier than one might expect at first glance (and our first try in the previous edition was inadequate). The approach in 3.1.0 allows the requisite induction to be limited to a subspraid $(\mathcal{W}, \mathcal{T}_{\#})$, which is far less restrictive than requiring induction on all of $(\mathcal{V}, \mathcal{T}_{\#})$.

A good counterexample to consider is given by the Kleene Tree K_{bar}. If we put $\mathcal{U} = \{[\![k]\!] \mid k \in K_{\text{bar}}\}$, then we see that $\mathcal{U}$ is an open cover of Cantor space in RUSS. But $\mathcal{U}$ is not an inductive open cover of Cantor space in RUSS, since K_{bar} is not an inductive cover of $\mathcal{C}$.

We do not delve into this definition since it is little used in this monograph, but one should study pointwise inductive open covers in more detail. This bears directly on the important question of how to 'inductivize' results from [Bis&Bri1985] and [Bri&Vîţ2006] to natural topology.

A.5.7 **Star-finitary metrization in INT** Our star-finitary metrization theorem in 4.0.8 is also intended as an example how intuitionistic results can be translated to our setting by inductivizing the definitions (as in chapter three).

However, the translation from [Waa1996] of the corresponding intuitionistic star-finitary metrization theorem was complicated by the discovery that the proof offered in [Waa1996] contains an error, specifically the proof of lemma 2.4.4. which corresponds to part of our Urysohn function lemma (**E**).

This error can be corrected by looking at (genetic) bars which represent the condition $A \# B$, as our proof here shows, but a straight translation was rendered impossible. This is the main reason that our proof here is more involved and longer than the proof in [Waa1996].

A.6 BIBLIOGRAPHY AND FURTHER READING REFERENCES

A.6.0 Further reading, research and researchers The bibliography given below contains the necessary research references for reading the monograph. It is also intended as a quick (therefore quite incomplete) overview of current research and researchers related to constructive topology. With current internet access to scientific publications, one should have no difficulty in finding other relevant research and researchers. Suggestions for improvement of the bibliography are welcome, through the website of the author (www.fwaaldijk.nl/mathematics.html, where his own few publications can also be found online).

A.6.1 Bibliography

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